Journal of Computational Mathematics, Vol.19, No.4, 2001, 433–444.

THE LIMITING CASE OF THIELE'S INTERPOLATING CONTINUED FRACTION EXPANSION[*1)]

Jie-qing Tan
(*Institute of Applied Mathematics, College of Science and CAD/CG Division, College of Computer & Information, Hefei University of Technology, Hefei 230009, China*)

Abstract

By means of the determinantal formulae for inverse and reciprocal differences with coincident data points, the limiting case of Thiele's interpolating continued fraction expansion is studied in this paper and given numerical example shows that the limiting Thiele's continued fraction expansion can be determined once for all instead of carrying out computations for each step to obtain each convergent as done in [3].

Key words: Continued fraction, Inverse difference, Reciprocal difference, Expansion.

1. Introduction

When we talk about the interpolation by polynomials, it is natural for us to have at heart the Lagrange interpolation, the Hermite interpolation and the Newton interpolation. Of these interpolants, the Newton interpolating polynomial is probably most favourite because of its advantages in carrying out computations and performing limits. As we know, a Newton interpolating polynomial is established on the basis of the divided differences, whose recursive calculation makes it possible that the Newton interpolant can be obtained by adding new support points one at a time after having interpolated the previous ones. Furthermore, the relation between divided differences and derivatives allows the support points in a Newton interpolant to coincide with one another, which is beyond the conditions required by the Lagrange polynomials. It is interesting to notice that the Newton's interpolating polynomial possesses its nonlinear counterpart, i.e., the Thiele's interpolating continued fraction, which is established in terms of inverse differences. Like divided differences, inverse differences are defined in a recursive manner and allow the occurrence of repeated support points. However, implementing the limit process for inverse differences is much more complicated than for divided differences. Although both the Thiele's method and Viscovatov's method are available for the computation of the limiting case of inverse differences, they expose the shortcomings that computations have to be carried out for each step to obtain each convergent.

In this paper, starting from the Newton-Padé approximants, we offer a new kind of determinantal representations for inverse and reciprocal differences which allow the coincidence of support points. A numerical example is given to support our argument that our method is more reliable in some cases than the Thiele's method and Viscovatov's method.

Suppose $f(x)$ is a function defined on a subset G of the complex plane and $X = \{x_i | i \in \mathbb{N}\}$ is the set of points belonging to G. Let

$$C_{ij} = \begin{cases} f[x_i, \ldots, x_j], & \text{for } i \le j \\ 0, & \text{for } i > j \end{cases} \tag{1.1}$$

where $f[x_i, \ldots, x_j]$ denotes the divided difference of the function $f(x)$ at points $x_i, \ldots, x_j$, and let

$$\omega_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1}) \tag{1.2}$$

* Received October 12, 1998; Final revised November 6, 1999.
1) Supported by the Foundation for Excellent Young Teachers of the Ministry of Education of China and in part by the Foundation for the Backbone Teachers of the Ministry of Education of China.

with $\omega_0(x) = 1$, then $f(x)$ can formally be expanded as the following Newton series

$$f(x) = \sum_{i=0}^{\infty} C_{0i}\omega_i(x) \tag{1.3}$$

It is known to all that one may find two polynomials

$$P_{m,n}(x) = \sum_{i=0}^{m} a_i\omega_i(x) \tag{1.4}$$

and

$$Q_{m,n}(x) = \sum_{i=0}^{n} b_i\omega_i(x) \tag{1.5}$$

such that

$$f(x)Q_{m,n}(x) - P_{m,n}(x) = \sum_{i\geq m+n+1} d_i\omega_i(x), \tag{1.6}$$

which is the very Newton-Padé approximation problem of order (m, n) for $f(x)$ (see [1], [3] or [9]). It is not difficult to verify that if the rank of the matrix

$$\begin{bmatrix} C_{0,m+1} & C_{1,m+1} & \cdots & C_{n,m+1} \\ C_{0,m+2} & C_{1,m+2} & \cdots & C_{n,m+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,m+n} & C_{1,m+n} & \cdots & C_{n,m+n} \end{bmatrix} \tag{1.7}$$

is maximal, then $P_{m,n}(x)$ and $Q_{m,n}(x)$ possess the following determinant representation

$$P_{m,n}(x) = C\begin{vmatrix} F_{0,m}(x) & F_{1,m}(x) & \cdots & F_{n,m}(x) \\ C_{0,m+1} & C_{1,m+1} & \cdots & C_{n,m+1} \\ C_{0,m+2} & C_{1,m+2} & \cdots & C_{n,m+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,m+n} & C_{1,m+n} & \cdots & C_{n,m+n} \end{vmatrix} \tag{1.8}$$

$$Q_{m,n}(x) = C\begin{vmatrix} \omega_0(x) & \omega_1(x) & \cdots & \omega_n(x) \\ C_{0,m+1} & C_{1,m+1} & \cdots & C_{n,m+1} \\ C_{0,m+2} & C_{1,m+2} & \cdots & C_{n,m+2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,m+n} & C_{1,m+n} & \cdots & C_{n,m+n} \end{vmatrix} \tag{1.9}$$

where

$$F_{i,j}(x) = \begin{cases} \sum_{k=i}^{j} C_{i,k}\omega_k(x), & \text{for } i \leq j \\ 0, & \text{for } i > j \end{cases} \tag{1.10}$$

and C is some nonzero normalization constant.

2. Thiele's Continued Fraction

Thiele's interpolating continued fraction is a continued fraction of the following form (see [7] and [8])

$$\varphi_0[x_0] + \frac{x - x_0}{|\varphi_1[x_0, x_1]} + \frac{x - x_1}{|\varphi_2[x_0, x_1, x_2]} + \cdots \tag{2.1}$$

where $\varphi_i[x_0, x_1, \ldots, x_i], i = 0, 1, 2, \ldots,$ are the inverse differences defined as follows

$$\begin{aligned} \varphi_0[x] &= f(x) \\ \varphi_1[x_0, x_1] &= \frac{x_1 - x_0}{\varphi_0[x_1] - \varphi_0[x_0]} \\ \varphi_k[x_0, x_1, \ldots, x_k] &= \frac{x_k - x_{k-1}}{\varphi_{k-1}[x_0, x_1, \ldots, x_{k-2}, x_k] - \varphi_{k-1}[x_0, x_1, \ldots, x_{k-2}, x_{k-1}]} \end{aligned} \tag{2.2}$$

for every $x_0, x_1, \ldots, x_k$ and x in G.