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Non-Semisimple Lie Algebras of Block Matrices and Applications to Bi-Integrable Couplings

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Abstract. We propose a class of non-semisimple matrix loop algebras consisting of 3×3 block matrices, and form zero curvature equations from the presented loop algebras to generate bi-integrable couplings. Applications are made for the AKNS soliton hierarchy and Hamiltonian structures of the resulting integrable couplings are constructed by using the associated variational identities.

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1 Introduction

For a given integrable system, integrable couplings are non-trivial larger systems which are still integrable and include the original integrable system as a sub-system. The concept of integrable couplings was systematically introduced in 1996 (see [16] for details), and since then it has been an attractive research topic of many publications (see, e.g., [7,8, 10,19,26–29,31,32]). A few methods of constructing integrable couplings have been developed, such as the perturbation method [8,15,16], enlarging spectral problems [10,11], and constructing new matrix loop Lie algebras [5,30]. Recently, a new class of non-semisimple matrix loop algebras was proposed in [21] for investigating nonlinear bi-integrable couplings.

In this paper, we will introduce 10 new classes of Lie algebras of 3×3 block matrices which can generate bi-integrable couplings.

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First, let us recall the problem of integrable couplings: for a given integrable system of evolution equations:

$$u_t = K(u), \tag{1.1}$$

where u is in some manifold M and K is a suitable C^∞ vector field on M, we look for an enlarged non-trivial integrable system which includes the original system as a subsystem. It is known that a change of the arrangement of equations in a system does not lose integrability of the system, and therefore we study how to construct an enlarged non-trivial system of evolution equations of the triangular form. Such a bi-integrable coupling of the system (1.1) is defined as follows [21]:

$$\begin{cases} u_t = K(u), \\ u_{1,t} = S_1(u,u_1), \\ u_{2,t} = S_2(u,u_1,u_2), \end{cases} \tag{1.2}$$

where u_1 and u_2 are new dependent variables, and S_1 and S_2 are vector fields depending on the indicated variables. We call this integrable system a nonlinear coupling if at least one of $S_1(u,u_1)$ and $S_2(u,u_1,u_2)$ is nonlinear with respect to the sub-vectors u_1,u_2 of dependent variables.

In this paper, we will introduce new non-semisimple Lie algebras of 3×3 block matrices in Section 2, and then in Section 3, we will describe a general scheme to construct bi-integrable couplings associated with the newly presented Lie algebras. Section 4 is devoted to applications to the AKNS hierarchy and mathematical structures that the resulting bi-integrable couplings possess, such as infinitely many symmetries, infinitely many conserved functionals, and bi-Hamiltonian structures.

2 Loop algebras of 3×3 block matrices

We seek for non-semisimple matrix Lie algebras, under which we can generate bi-integrable couplings of an integrable system (1.1) by using the zero curvature equation. First, we look for matrix algebras consisting of 3×3 block matrices of the form

$$M(A_1,A_2,A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & \sum_{i=1}^{3}\alpha_{1,i}A_i & \sum_{i=1}^{3}\alpha_{2,i}A_i \\ 0 & 0 & \sum_{i=1}^{3}\alpha_{3,i}A_i \end{bmatrix},$$

where $\alpha_{i,j}$, $1\le i,j\le 3$ are constants to be determined. The reason why we choose these triangular type block matrices is that Lax pair [6] matrices U and V of triangular types will help generate bi-integrable couplings. Thus in the next step, we want to classify classes of such matrices which form matrix Lie algebras under matrix commutator

$$[U,V] := UV - VU. \tag{2.1}$$