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SUPERCONVERGENCE OF TETRAHEDRAL LINEAR FINITE ELEMENTS

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Abstract. In this paper, we show that the piecewise linear finite element solution u_h and the linear interpolation u_I have superclose gradient for tetrahedral meshes, where most elements are obtained by dividing approximate parallelepiped into six tetrahedra. We then analyze a post-processing gradient recovery scheme, showing that the global L^2 projection of ∇u_h is a superconvergent gradient approximation to ∇u.

Key Words. superconvergence, finite element methods, tetrahedral elements, post-processing

1. Introduction

Superconvergence of the gradient for the finite element approximation for second order elliptic boundary value problems and gradient recovery schemes have been an active research topic; see, for example, Babuška and Strouboulis [1], Chen and Huang [8], Lin and Yan [12], Wahlbin [13] and Lakhany, Marek, and Whiteman [11] for overviews of this field. Recently Bank and Xu [2, 3] have developed some new techniques and obtained some new superconvergence results for linear finite element elements on two dimensional triangular meshes. The goal of this paper is to extend their results to three dimensions, namely to linear tetrahedral finite element.

The model problem that we study in this paper is

$$
\begin{aligned}
&-\nabla \cdot (\mathcal{D}(x)\nabla u) = f,\ x \in \Omega\\
&u = 0,\ x \in \partial\Omega.
\end{aligned}
$$

Here $\mathcal{D}(x)$ is a 3×3 symmetric matrix function in $(L^\infty(\Omega))^{3\times 3}$ and uniformly positive definite. For simplicity, we assume f is smooth enough and Ω is a polyedr in $\mathbb{R}^3$ partitioned into a quasiuniform triangulation $\mathcal{T}_h$ with mesh size $h \in (0, 1)$. Let $\mathcal{V}_h \subset H_0^1(\Omega)$ be the corresponding finite element space of continuous piecewise linear functions associated with $\mathcal{T}_h$, and let $u_h \in \mathcal{V}_h$ be the finite element solution of the above second order elliptic boundary value problem.

Unlike in the two dimensional case, superconvergence results in three dimensions are relatively rare [7, 9, 10, 5]. The difficulty is partially due to the loss of symmetry in three dimensions [4]. In this paper, we only deal with a special triangulation of which most elements are obtained by dividing each $O(h^2)$ parallelepiped into six tetrahedra (see Section 3 for details). For this kind of triangulation, we numerically

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observed that superconvergence occurs for linear elements, due to the cancellation of the lowest order terms in some asymptotic expansion of the local error. It is, however, difficult to combine elementwise error estimates together, since the normal component of the gradients of the test functions is discontinuous. Thus we follow the new approach in [2] to derive some expressions for the element error that involve only the tangential derivative of the test function on the edges.

Our first result is that the gradient of the finite element approximation u_h is superclose to the gradient of the piecewise linear interpolant u_I of the solution u. More precisely, we have

$$|u_h - u_I|_{1,\Omega} \lesssim h^{1+\min(\sigma,1)} \|u\|_{3,\infty,\Omega}. \tag{1}$$

Estimate (1) holds on quasi-uniform meshes, where most elements are obtained by dividing each $O(h^2)$ parallelepiped into six tetrahedra except for a region of size $O(h^{2\sigma})$; see Section 3 for details.

The estimate (1) is known in the literature for the special case $\sigma = \infty$ [7, 9, 10]. Recently Brandts and Křížek [5] extend the results of [7, 9, 10] for tetrahedra into arbitrary n- simplex. Our new estimate (1) is a significant generalization, since firstly, our analysis is based on local identities for each element and thus, it is straightforward to extend our results to meshes in which an $O(h^{1+\alpha})$ (instead of $O(h^2)$) approximate local symmetry property holds for most patches of edges. Second, the relaxation parameter σ makes our analysis to work for more general meshes, especially for domains with unstructured boundaries.

Based on the superconvergence results, one can construct schemes to get better approximations of ∇u; see for example, [16, 17, 14, 15] and [6]. The second major component of this work is a superconvergent approximation to ∇u by a gradient recovery procedure. In Section 4, we show that

$$\|\nabla u - Q_h \nabla u_h\|_{0,\Omega} \lesssim h^{1+\min(\sigma,1/2)} \|u\|_{3,\infty,\Omega}, \tag{2}$$

where Q_h is the L^2 projection to $\mathcal{V}_h^3$. As remarked in [2], both the superconvergence and gradient recovery results can be generalized to a more general non-self-adjoint and possibly indefinite problem.

The rest of this paper is organized as follows. We introduce some notation and technical identities for our analysis in Section 2. We prove the estimate (1) and (2) in Section 3 and Section 4 respectively.

2. Local Error Expansion

In this section we shall derive some useful identities for our analysis. The key identity is contained in Lemma 2.4, which is a generalization of the integral formulas of rectangular elements [12] and triangular elements [2] in two dimensions to tetrahedral elements in three dimensions.

Let τ be a tetrahedron in $\mathbb{R}^3$, with vertices $\{\mathbf{p}_k\}_{k=1}^4$ and the corresponding nodal basis functions (barycentric coordinates) $\{\varphi_k\}_{k=1}^4$. We assume that $\mathbb{R}^3$ has the orientation given by the right-hand rule and τ has the induced orientation. Let F_k denote the surface opposite vertex $\mathbf{p}_k$ with the induced orientation and $\mathbf{n}_k$ the unit outward normal vectors of F_k. We also use the symbol $\triangle_{klm}$ to denote the face with vertices $\mathbf{p}_k, \mathbf{p}_l$, and $\mathbf{p}_m$. If the orientation of $\triangle_{klm}$, given by the order of k, l, m, coincides with the induced orientation from τ, we say $\triangle_{klm}$ has the consistent orientation with τ. Let $\mathbf{e}_{ij}$ denote the oriented edges of element τ from $\mathbf{p}_i$ to $\mathbf{p}_j$ and $\mathbf{t}_{ij}$, d_{ij} the corresponding unit tangent vectors and length, respectively (see Fig 1). Let θ_{kl} be the angle between $\mathbf{t}_{kl}$ and the supporting plane of F_l. In