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GLOBAL SUPERCONVERGENCE FOR OPTIMAL CONTROL PROBLEMS GOVERNED BY STOKES EQUATIONS

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Abstract. In this paper, the global superconvergence analysis for the finite element approximation of the distributed optimal control governed by Stokes equations is discussed. For the control, a global superconvergence result is derived by applying patch recovery technique. For the state and the co-state, the global superconvergence results are derived by applying some postprocessing techniques for the bilinear-constant scheme over the uniform rectangular meshes. Based on the global superconvergence analysis, recovery type a posteriori error estimates are derived. It is shown that the recovery type a posteriori error estimators provided in this paper are asymptotically exact if the conditions for the superconvergence are satisfied.

Key Words. optimal control, Stokes equations, finite element approximation, global superconvergence, recovery, a posteriori error estimate.

1. Introduction

Flow control problems are crucial to many engineering applications. Extensive research has been carried out on various theoretical aspects of flow control problems, see, for example, [1], [6], [8], [10], [21], [22], [27], [30]. It is obvious that efficient numerical methods are essential to successful applications of flow control. It is well known that the finite element method is undoubtedly the most widely used numerical method in computing optimal control problems, including flow control problems. Systematic introductions to the finite element method for PDEs and optimal control problems can be found in, for example, [3], [11], [28], and [30]. There have been extensive theoretical studies of finite element approximation for various optimal control problems. For instance, a priori error estimates of finite element approximation were established long ago for the optimal control problems governed by linear elliptic and parabolic state equations; see, for example, [5], [13], and [26]. Furthermore, finite element approximation of some flow control has been studied, and a priori error estimates have been established; see [8], [9], [10], and [12]. A posteriori error estimates of finite element approximation were derived for the optimal control problems governed by Stokes equations and for convex boundary control problems, see, i.e., [2], [23], [24], [25].

In recent years, superconvergence for finite element solutions has been an active research area in numerical analysis. The main objective for superconvergence is

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to improve the existing approximation accuracy by applying certain postprocessing techniques which are easy to implement. For the stationary Stokes problem, many finite element schemes which satisfy the Babuška-Brezzi condition have been introduced in [7]. It has been found that there exits a potential for high accuracy or superconvergence for several finite element schemes when the exact solution is smooth enough and the mesh is sufficiently regular; see, for example, [16], [17], [18], [29]. A principle technique for the proof of the global technique is the integral identity technique which has proven to be an efficient tool for the superconvergence analysis of rectangular finite elements (cf. [15]). For the distributed convex optimal control problems governed by elliptic equations, some superconvergence results have been established by applying recovery operators (see, i.e., in [4], [14]).

In this paper, by means of the techniques used in [14] and [29], the global superconvergence for the control problems governed by Stokes equations is discussed. It is shown that if the solution is smooth enough, the mesh for the state and the co-state is the uniform rectangular mesh and the bilinear-constant scheme is adopted for the state and co-state equations, the global superconvergence for the control, state and co-state can be proved. Based on the supercovergence analysis, recovery type a posteriori error estimators are provided.

The outline of this paper is as follows: In Section 2, we provide a weak form for the distributed control problem governed by Stokes equation and its finite element approximation scheme. In Section 3, a global superconvergence result for the control $\mathbf{u}$ is derived by applying recovery operator and the supercovergence analysis technique. Moreover, the global superconvergence results for the state $\mathbf{y}$ and the co-state $\mathbf{p}$ (also r and s) are derived by applying the integral identity technique in Section 4. In the Section 5, based on the global superconvergence analysis provided in Sections 3 and 4, the recovery type a posteriori error estimate is discussed. In the last section, we discuss briefly some possible future work.

Let Ω and $\Omega_{\mathbf{U}}$ be two bounded open sets in R^2 with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_{\mathbf{U}}$, respectively. In this paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|.\|_{m,q,\Omega}$ and seminorm $|.|_{m,q,\Omega}$. We shall extend these (semi) norms to vector functions whose components belong to $W^{m,q}(\Omega)$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$ with the norm $\|.\|_{m,\Omega}$ and the seminorm $|.|_{m,\Omega}$. In addition, c or C denotes a general positive constant independent of h.

2. Finite element approximation of optimal control problems

In this section, we discuss the finite element approximation of distributed convex optimal control problems governed by the Stokes equations. Let $\mathbf{Y} = (H_0^1(\Omega))^2, \mathbf{U} = (L^2(\Omega_{\mathbf{U}}))^2, \mathbf{H} = (L^2(\Omega))^2$, and $Q = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0\}$. The state space and the control space will be $\mathbf{Y} \times Q$ and $\mathbf{U}$, respectively. Let B be a linear continuous operator from $\mathbf{U}$ to $\mathbf{H}$, let g be a strictly convex functional which is continuously differentiable on $\mathbf{H}$, and let $\mathbf{K}$ be a closed convex set in the control space $\mathbf{U}$ such that

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{U} : \mathbf{v} \geq 0\}.$$

We further assume that the functional $g(\cdot)$ is bounded below.