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SUPERCONVERGENCE OF LEAST-SQUARES MIXED FINITE ELEMENTS

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Abstract. In this paper we consider superconvergence and supercloseness in the least-squares mixed finite element method for elliptic problems. The supercloseness is with respect to the standard and mixed finite element approximations of the same elliptic problem, and does not depend on the properties of the mesh. As an application, we will derive more precise a priori bounds for the least squares mixed method. The superconvergence may be used to define a posteriori error estimators in the usual way. As a by-product of the analysis, a strengthened Cauchy-Buniakowskii-Schwarz inequality is used to prove the coercivity of the least-squares mixed bilinear form in a straight-forward manner. Using the same inequality, it can moreover be shown that the least-squares mixed finite element linear system of equations can basically be solved with one single iteration step of the Block Jacobi method.

Key Words. least squares mixed elements, supercloseness, superconvergence

1. Introduction

Superconvergence in finite element methods is an important topic in current research, as is reflected in the references in the classical overview paper [19] but also in the proceedings [20] and of course this issue of this journal. In the past decade, much progress has been made. On the one hand, the so-called Chinese school [12, 29, 30] has made progress in developing suitable interpolants of the exact solution of a PDE to which its finite element approximation is *superclose*. This strategy became necessary since results in [21] (and earlier work by the same author) showed that the nodal interpolant often lacks this property, in particular for n-simplicial elements in dimension $n \geq 2$ of degree d with $d > n$. On the other hand, the so-called *patch-recovery technique* [28, 26, 27] allows for superconvergence on irregular meshes at the cost of additional computations on a patch of elements surrounding an element. Finally, progress has also been made in proving (and, in fact, disproving) localized bounds [14, 15, 23, 24, 25] for standard and mixed finite element methods.

1.1. Least squares mixed finite elements.

In this paper we turn our attention to supercloseness and superconvergence in least-squares mixed finite element methods [11, 22] for elliptic equations. These methods aim to provide approximations for the potential and the flux separately, just as mixed finite element methods. The difference is that instead of posing a Ritz-Galerkin condition to select approximations from the subspaces, which results in a saddle-point problem that is not trivial

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[9, 10] to discretize, it employs a least-squares approach. Just as in the standard Galerkin method, this leads to a symmetric coercive bilinear form and straightforward discretization.

A drawback of the least-squares mixed finite element method is that the errors in both the potential and the flux influence one another; as a result, the well-known Lemma by Céa is only able to yield a bound for the largest error of the two. It may however be the case that one of the two errors is of higher order than the other. To prove that in some situations this is indeed the case, one needs to rely on other techniques. In [5] it was proposed to use supercloseness of the least-squares mixed finite element approximations to well-known and well-defined reference functions from the approximating spaces in order to give separate results for both the potential and the flux.

1.2. Outline of this paper. We start in Section 2 with defining our model problem and fix our notations for Sobolev spaces and norms, in particular for some weighted norms on product spaces. In Section 2.2, we recall the strengthened Cauchy-Buniakowskii-Schwarz (CBS) inequality from [6] and put it in a slightly more general context. In Section 2.3 we describe the least-squares mixed finite element method for our model problem and give a one-line proof of the coercivity of the associated bilinear form. Due to the strengthened CBS inequality, block-diagonal preconditioning of the linear system results in a condition number of the preconditioned matrix that is bounded uniformly in the stepsize; as an illustration, we prove separately that the block-Jaboci method (which is equivalent to the block-diagonally preconditioned Richardson iteration) has convergence factor γ when measured in the appropriate norm. Then, in Section 3, we turn to the application of supercloseness to derive a priori bounds for the separate variables that improve the standard bounds by Céa's Lemma in case both approximating spaces have different approximation quality. Finally, we briefly discuss superconvergence by post-processing as a consequence of the supercloseness.

2. Preliminaries

As our model serves the following second order elliptic problem. Given $f \in H^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a convex polytope, find $u \in H_0^1(\Omega)$ such that

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where A is uniformly symmetric positive definite with Lipschitz continuous coefficients and with eigenvalues in the interval $[\beta^2, \beta^{-2}]$ for some $\beta \in (0, 1]$. The formulation of (1) as a system of first-order equations lies at the basis of the least-squares mixed finite element method. This formulation is to find functions $u \in H_0^1(\Omega)$ and $\mathbf{p} \in \mathbf{H}(\operatorname{div}; \Omega)$ such

$$\mathbf{p} = -A\nabla u \text{ in } \Omega, \quad \operatorname{div} \mathbf{p} = f \text{ in } \Omega. \tag{2}$$

Since the spaces $H_0^1(\Omega)$ and $\mathbf{H}(\operatorname{div}; \Omega)$ play a central part in the analysis, we will derive a useful but nevertheless simple result that involves both of them. First however some notations.

2.1. Weighted Sobolev norms and other notations. We use standard notations for Sobolev spaces and their norms and semi-norms; the L_2-norm and inner product we denote by $|\cdot|_0$ and $(\cdot,\cdot)_0$. Additional to the usual norms on $\mathbf{H}(\operatorname{div}; \Omega)$