Journal of Computational Mathematics, Vol.18, No.5, 2000, 457–466.

ON THE CONVERGENCE OF KING-WERNER ITERATION METHOD IN BANACH SPACE*1)

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Abstract

In this paper , a Kantorovitch-Ostrowski type convergence theorem and an error estimate of $\frac{\|f'(z_0)^{-1}f(x_{n+1})\|}{\|f'(z_0)^{-1}f(x_n)\|}$ using the information of higher derivatives at the center between initial points for King-Werner iteration method in Banach space are established.

Key words: Information at the center between initial points, King-Werner iteration method, Convergence, Error estimate.

1. Introduction

Let

$$f(x) = 0 \tag{1.1}$$

where $f : X \to Y$ is a nonlinear operator which maps Banach space X into Banach space Y. The well-known iteration methods for solving (1.1) are the Newton method and very kinds of its improvement methods. One of them is the so called King-Werner method defined by

$$kw(P, x_0, y_0): \quad \begin{cases} z_n = \dfrac{x_n + y_n}{2} \\ x_{n+1} = x_n - f'(z_n)^{-1} f(x_n) \\ y_{n+1} = x_{n+1} - f'(z_n)^{-1} f(x_{n+1}) \end{cases} \quad \forall n \in N_0, \tag{1.2}$$

which is established by King in [7], Werner in [12] in different formulas, respectively. It is interesting that the method (1.2) is of order $1 + \sqrt{2}$ with the same function computation cost and two times combination cost as that of Newton method. Define

$$\omega(x, z) = x - f'(z)^{-1} f(x),$$

then (1.2) can be rewritten as

$$kw(P, x_0, y_0): \quad \begin{cases} z_n = \dfrac{x_n + y_n}{2} \\ x_{n+1} = \omega(x_n, z_n) \\ y_{n+1} = \omega(x_{n+1}, z_n) \end{cases} \quad \forall n \in N_0. \tag{1.3}$$

* Received January 26, 1997.
1)Partial Supported by the Natural Science Foundation of Zhejiang Province.

There are a number of papers concerning the convergence of Newton method and its improvement methods under the condition of Kantorovitch theorem or relatively close ones (e.g.[2],[6],[8]-[11],[13] etc.). In [4] [5], Kantorovich type convergence theorems and estimates of Newton method and two Newton-like methods using higher derivatives information are proved, respectively, if f has higher derivatives, though they are not used in iteration process. The idea of using higher derivatives at initial points is also used for Halley method in [14], and for a class of parameter based Chebyshev-Halley type methods in [3], where the higher derivatives are used in iteration process.

In this paper, a convergence theorem of Kantorovitch-Ostrowski type using higher derivatives at the center between initial points for King-Werner method (1.2) is established. Also, an error estimate of the decreasing speed of $\dfrac{\|f'(z_0)^{-1}f(x_{n+1})\|}{\|f'(z_0)^{-1}f(x_n)\|}$ is obtained. We put forth the main results in §2 and give the proofs and an example in §3.

2. Main Results

Define $\overline{O(z,t)} = \{x \in X | \|x-z\| \le t\}$, $O(z,t) = \{x \in X | \|x-z\| < t\}$, where $z \in X$.

Theorem 2.1. *Let X, Y be Banach spaces, $f : X \to Y$ have first- and second-order Frechet derivatives, which are bounded linear operators from X to Y and X to $L(X,Y)$, respectively. Suppose $x_0, y_0 \in D \subset X$, a convex subset of X, $z_0 = \frac{x_0+y_0}{2}$, and*

$$\begin{aligned}
&\|x_0 - y_0\| \le \tau, \qquad \|x_1 - y_0\| \le \eta,\\
&\|f'(z_0)^{-1}f''(z_0)\| \le \gamma,\\
&\|f'(z_0)^{-1}[f''(x) - f''(y)]\| \le K\|x-y\| \qquad \forall x, y \in D.
\end{aligned}$$

If $\overline{O(z_0, t^* - \frac{\tau}{2})} \subset D$,

$$3(\eta + \psi(\tau))\gamma \le \frac{\gamma + 2\sqrt{\gamma^2 + 2K}}{\gamma + \sqrt{\gamma^2+2K} + K} \tag{2.1}$$

and

$$\frac{K}{2}(\frac{\tau}{2} + \eta)^2 + \gamma(\frac{\tau}{2} + \eta) - 1 < 0, \tag{2.2}$$

where $\psi(\tau) = \frac{1}{48}K\tau^3 - \frac{1}{8}\gamma\tau^2 + \frac{1}{2}\tau$, then

i) *the sequence $kw(f; x_0, y_0)$ defined by (1.2) starting from x_0, y_0 converges to the unique solution of $f(x)$ in $\overline{O(z_0, t^* - \frac{\tau}{2})} \cup O(z_0, t^{**} - \frac{\tau}{2}) \cap D$, where $0 < t^* \le t^{**}$ are two positive zeros of the polynomial*

$$\phi(t) = \frac{K}{6}(t - \frac{\tau}{2})^3 + \frac{1}{2}\gamma(t - \frac{\tau}{2})^2 - (t - \frac{\tau}{2}) + \frac{\tau}{2} + \eta - \frac{\gamma}{8}\tau^2 + \frac{K}{48}\tau^3. \tag{2.3}$$

ii)

$$\|x_n - x^*\| \le t^* - t_n \qquad \|x^* - y_n\| \le t^* - s_n \qquad \forall n \in N_0$$