Journal of Computational Mathematics, Vol.18, No.6, 2000, 571–586.

SEMIDISCRETIZATION IN SPACE OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH BLOW-UP OF THE SOLUTIONS*

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Abstract

Semidiscretization in space of nonlinear degenerate parabolic equations of non-divergent form is presented, under zero Dirichlet boundary condition. It is shown that semidiscrete solutions blow up in finite time. In particular, the asymptotic behavior of blowing-up solutions, is discussed precisely.

Key words: Semi-discrete problem, Blow-up of solutions, Blow-up rate, Blow-up set, Limiting profile.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider the following initial boundary value problem :

$$\text{(P1)} \qquad \begin{cases} u_t = u^\delta(\triangle u + \mu u), & x \in \Omega, t > 0, & (1)\\ u(x,t) = 0, & x \in \partial\Omega, t > 0, & (2)\\ u(x,0) = u_0(x), & x \in \Omega, & (3) \end{cases}$$

where δ, μ are positive constants and $u_0(x)$ is a nonnegative bounded continuous function on $\bar{\Omega}$.

When $N = 1$ and $\delta = 2$, the problem arises in a model for the resistive diffusion of a force-free magnetic field in a plasma confined between two walls in one dimension (see [5], [8], [9], [10] and [14]). Equation (1) also describes the evolution of the curvature of a locally convex plane curve, and it has been studied in [2] and [6] under periodic boundary condition.

A. Friedman and B. McLeod [5] considered (P1) in the case $\delta = 2$ and $\mu = 1$. They showed that the behavior of solutions depends on the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet problem for the Laplacian on the domain Ω. If $\lambda_1(\Omega) > 1$, then there exists a unique global solution which tends to zero as $t \to \infty$. If $\lambda_1(\Omega) < 1$, then

* Received December 7, 1996.

there exists a positive constant T such that we have a unique solution in $0 < t < T$, which blows up as $t \uparrow T$. They also showed that the blow-up set has positive Lebesgue measure. In particular, when $N = 1$ and initial data u_0 satisfies $u_0(-x) = u_0(x)$ and $u_{0xx} + u_0 \geq 0$, they showed that the blow-up set S is exactly $S = \{-\pi/2 \leq x \leq \pi/2\}$. Qi [12] discussed the Cauchy problem for (1) and (3) with $0 < \delta < 2$. For the case $\delta > 1$, M. Wiegner [15] studied the existence and uniqueness of smooth positive solutions and gave an upper bound of the blow-up time for the positive initial data. When $N = 1$ and $\delta > 0$, K. Anada, I. Fukuda and M. Tsutsumi [1] got precise information on the blow-up set and asymptotic behavior near the blow-up time. When $N \geq 2$, in [13] we have obtained the detailed results on the blow-up sets and asymptotic behavior of solutions of the problem (P1) with radially symmetric positive initial data. In [7], we solved this problem numerically by using a finite difference scheme with a variable time increment with suitable control and showed numerical results for symmetric and non-symmetric blowing-up solutions.

We consider the following two different levels in discretization of the problem:

Step 1. First, the problem (P1) is discretized in space. We use finite difference method as this discretization and get an ordinary differential system in a finite dimension. We call it "semidiscrete problem."

Step 2. Next, we discrete the semidiscrete problem in time by finite difference method. In order to compute a blowing-up solution suitably, we have to apply some control to time increment. (See [7].) We call this "variable time increment method". This idea is seen in [11], [3] and [4], in which they use variable time increment method for semilinear parabolic equations.

In this paper we consider the semidiscrete problem of (P1) for rectangle domain $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$ and analyze properties of solutions of semidiscrete problem. Of course, we can get the same results in higher dimension. We prove the blow-up of solutions of the semidiscrete problem for $\delta > 0$ and obtain lower and upper rates of blowing-up solution. We also get the lower and upper estimates of blow-up time and discuss the asymptotic behavior of solutions near the blow-up time.

2. Semidiscretization in Space of the Problem

First of all, in order to analyze a problem on a grid point set, we define a grid point set R_h with mesh size h (> 0) by $R_h = \{x_m \in \mathbb{R} | x_m = hm, m \in \mathbb{Z}\}$. Let M and N be positive integers and take $h_x = a/M$ and $h_y = b/N$. Then we now define Ω_h, which is the discretization of Ω, as the following:

$$\Omega_h = \{(x_m, y_n) \in R_{h_x} \times R_{h_y} | (x_m, y_n) \in \Omega\}.$$

Next, we introduce the following terms and notion to express the statements precisely.

Definition 2.1. (*neighboring grid points, neighboring set*) (x_{m+1}, y_n), (x_{m-1}, y_n), (x_m, y_{n+1}) *and* (x_m, y_{n-1}) *are called neighboring grid points of* (x_m, y_n). *The neighbor-*