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A POSTERIORI ERROR ESTIMATOR FOR SPECTRAL APPROXIMATIONS OF COMPLETELY CONTINUOUS OPERATORS

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Abstract. In this paper, we study numerical approximations of eigenvalues when using projection method for spectral approximations of completely continuous operators. We improve the theory depending on the ascent of $T - \mu$ and provide a new approach for error estimate, which depends only on the ascent of $T_h - \mu_h$. Applying this estimator to the integral operator eigenvalue problems, we obtain asymptotically exact indicators. Numerical experiments are provided to support our theoretical conclusions.

Key Words. completely continuous operators, projection method, eigenvalues, a posteriori error estimates

1. Spectral Approximations of Completely Continuous Operators

In this paper, we assume that X is a separable reflexive Banach space or a separable Hilbert space, $\|\cdot\|$ and $<\cdot,\cdot>$ are the norm and the adjoint pair in X, respectively. Let S^h be a sequence of finite dimensional spaces such that

$$S^{h_1} \subset S^{h_2} \quad \forall h_2 < h_1; \qquad \overline{\bigcup_{h>0} S^h} = X.$$

We will consider a completely continuous operator $T : X \to X$ and a family of finite ranked operators $T_h : X \to X$, such that

$$\|T_h - T\| \to 0 \quad (h \to 0).$$

Consider the operator eigenvalue problem: Find $\mu \in C$, $0 \neq u \in X$, such that

$$Tu = \mu u. \tag{1}$$

Also consider its discrete scheme: Find $\mu_h \in C$, $0 \neq u_h \in S^h$, such that

$$T_h u_h = \mu_h u_h. \tag{2}$$

Let μ be an eigenvalue of T with algebraic multiplicity m, let E be the spectral projection associated with T and μ, and let E_h be the spectral projection associated with T_h and the eigenvalues of T_h which converge to μ. Similarly, let E^* and E_h^* be spectral projections associated with the adjoint T^* of T and the adjoint T_h^* of T_h, respectively. Moreover, denote $R(E)$, $R(E_h)$, $R(E^*)$, and $R(E_h^*)$ the image spaces of E, E_h, E^*, and E_h^*, respectively.

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In [4], Chatelin has proved that there exist m eigenvalues of T_h (including multiplicity) $\mu_{1,h},\ \mu_{2,h},\ \ldots,\ \mu_{m,h}$ converging to μ and $\mu_{1,h}, \mu_{2,h},\ \ldots,\ \mu_{m,h}$ are not necessarily equal, neither are the ascent of μ and that of $\mu_{i,h}$. In addition, the abstract error estimates of approximate eigenvalues and eigenfunctions have been studied since 1964 by Babuška, Bramble, Chatelin, Grigorieff, Lemordant, Osborn, Stummel, Vainikko, etc. A systematic summarization is found in [1]. We will need the following lemmas [1].

Lemma 1. *There is a constant c independent of h, such that*

$$\theta(R(E), R(E_h)) \leq c \cdot \|(T - T_h)\mid_{R(E)}\| \tag{3}$$

for small h, where $(T - T_h)\mid_{R(E)}$ denotes the restriction of $T - T_h$ to $R(E)$.

Lemma 2. *Let $\varphi_1, \cdots, \varphi_m$ be any basis for $R(E)$, and $\varphi_1^*, \cdots, \varphi_m^*$ be the dual basis for $R(E^*)$. We define $\bar{\mu}_h = \frac{1}{m} \cdot \sum_{j=1}^{m} \mu_{j,h}$, then there is a constant c independent of h, such that*

$$\begin{aligned} \mid \mu - \bar{\mu}_h \mid \ &\leq\ \frac{1}{m}\sum_{j=1}^{m} \mid< (T - T_h)\varphi_j, \varphi_j^* >\mid \\ &+\ c \cdot \|(T - T_h)\mid_{R(E)}\| \|(T^* - T_h^*)\mid_{R(E^*)}\|. \end{aligned} \tag{4}$$

Lemma 3. *Let α be the ascent of $\mu - T$. Let $\varphi_1, \cdots, \varphi_m$ be any basis for $R(E)$, and $\varphi_1^*, \cdots, \varphi_m^*$ be the dual basis for $R(E^*)$. Then there is a constant c, such that*

$$\begin{aligned} \mid \mu - \mu_{j,h} \mid \ &\leq\ c\{\sum_{i,k=1}^{m} \mid< (T - T_h)\varphi_i, \varphi_k^* >\mid \\ &+\ \|(T - T_h)\mid_{R(E)}\| \|(T^* - T_h^*)\mid_{R(E^*)}\|\}^{\frac{1}{\alpha}} \\ &\quad (j = 1, 2, \cdots, m). \end{aligned} \tag{5}$$

Lemma 4. *Let μ_h be an eigenvalue of T_h such that $\lim_{h\to 0} \mu_h = \mu$. Suppose for each h, u_h is a unit vector satisfying $(\mu_h - T_h)^k u_h = 0$ for some positive integer $k \leq \alpha$. Then for any integer j with $k \leq j \leq \alpha$, we have*

$$\|u_h - P_j u_h\| \leq c \cdot \|(T_h - T)\mid_{R(E)}\|^{\frac{j-k+1}{\alpha}}, \tag{6}$$

where P_j is the projection on $N((\mu - T)^j)$ along M_j . M_j is a closed subspace of X, such that $X = N((\mu - T)^j) \oplus M_j$.

These Lemmas provide a foundation of the spectral approximate theory for completely continuous operators. We can establish *a prior* error estimates of finite element solution for differential operators and integral operators by using these Lemmas. However, we shall note that (5) and (6) depend on the ascent α of $T - \mu$, which is very difficult to determine for non-self adjoint eigenvalue problems. Furthermore, the value of the constant c is unknown in (5) and (6). So, it is inconvenient to obtain *a posteriori* error estimates.

Since Babuška and Rheinboldt published the first paper on *a posteriori* error estimates of finite element methods [2], many developments have been made in this subject. In [6], an abstract error estimate has been presented, which gives *a posteriori* error estimates to finite element approximations for self-adjoint compact operator eigenvalue problems. In the cuurent paper, we will present an abstract