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THE FULL DISCRETE DISCONTINUOUS FINITE ELEMENT ANALYSIS FOR FIRST–ORDER LINEAR HYPERBOLIC EQUATION*1)

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Abstract

In this paper, the full discrete discontinuous Galerkin finite element method to solove 2–dimensional first–order linear hyperbolic problem is considered. Two practical schemes, Euler scheme and Crank–Nicolson scheme, are constructed. For each of them, the stability and error estimation with optimal order approximation is established in the norm stronger than L^2–norm.

Key words: Hyperbolic equation, Discontinous F.E.M., Euler scheme

1. Introduction

Let $\bar{\Omega}$ be a bounded domain in $\mathbf{R}^2$ with piecewise smooth boundary $\partial\Omega$, $[0,T]$ be a time interval. Consider the first-order hyperbolic problem as following

$$\frac{\partial u}{\partial t}+\beta(x,t)\cdot\nabla u+\sigma(x,t)u=f(x,t),\quad t\in(0,T],x\in\tilde{\Omega}(t), \tag{1.0a}$$
$$u(x,t)=g(x,t),\quad t\in[0,T],x\in\partial\Omega_-(t), \tag{1.0b}$$
$$u(x,t)=u_0(x),\quad x\in\Omega. \tag{1.0c}$$

where $\nabla u=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y})$, $\beta(x,t)=(\beta_1(x,t),\beta_2(x,t))$, $\partial\Omega_-(t)=\{x\in\partial\Omega:\beta(x,t)\cdot\gamma<0\}$, $\gamma(x)$ is the outward unit normal to $\partial\Omega$; $\tilde{\Omega}(t)=\bar{\Omega}\backslash\partial\Omega_-(t)$. As usual, $\partial\Omega_-(t)$ is refered to as inflow boundary at time t, and $\partial\Omega_+(t)=\partial\Omega\backslash\partial\Omega_-(t)$ is called outflow boundary at time t.

For simplicity in finite element analysis, suppose that boundary $\partial\Omega_-(t)$ is independent of t. Thus for all $t\in(0,T]$ we can write

$$\partial\Omega_-(t)\equiv\Gamma_-,\partial\Omega_+(t)\equiv\Gamma_+,\tilde{\Omega}(t)=\bar{\Omega}\backslash\Gamma_-\equiv\Omega^\star$$

and problem (1.0) can be written as

$$\frac{\partial u}{\partial t}+\beta(x,t)\cdot\nabla u+\sigma(x,t)u=f(x,t),\quad (x,t)\in\Omega^\star\times(0,T], \tag{1.1a}$$
$$u(x,t)=g(x,t),\quad (x,t)\in\Gamma_-\times[0,T], \tag{1.1b}$$

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$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.1c}$$

We shall consider the full discrete discontinuous Galerkin method for problem (1.1).

Set $D = \Omega \times (0,T]$, $L^p(0,T;X) \equiv L^p(X)$, $p = 2, +\infty$, where X is a Banach space. Assume that $\beta_i \in L^\infty(C^1(\bar{\Omega}))$, $i = 1,2$; $\sigma \in L^\infty(L^\infty(\Omega))$, $f \in L^2(L^2(\Omega))$, $g \in L^2(L^2(\Gamma_-))$; $u_0 \in L^2(\Omega)$.

Discontinuous Galerkin (DG) method is an explict method with good stability and satisfactory accuracy, thus it has become to be an effictive procedure to solve first-order hyperbolic problems. DG method was proposed by P. Lesaint and P.R. Raviart in 1978 ([1]), then it was developed by C. Johnson, G.R. Richart et al.[2–4]. In principle, we can use the DG method based on space-time finite element discretization for domain $\bar{\Omega} \times [0,T]$ to solve Problem (1.1), but in this case, we must solve a series of discretization problems defined on 3–dimensional subdomain $\bar{\Omega} \times [t_{n-1}, t_n]$, $n = 1, 2, \cdots$; As compared with full discrete Galerkin method, the computational scale of DG method is larger and the computing program is more complex.

In order to overcome the weakness of DG method, we now present a simplified DG method for time-dependent Problem (1.1), full discrete discontinuous Galerkin (FDDG) method, that is, using DG discretization only in space variables and using finite difference discretization in time variable t.

One can imagine that FDDG scheme possesses similar stability and convergence resultes with the DG scheme (based on space-time finite element). In fact, the theoritical analysis for FDDG scheme is more complex than that of DG scheme because of the non-uniform processing in time and space variables. It seems to us so far that there has been no paper to establish complete analysis for FDDG scheme of Problem (1.1).

In section 2 two practical FDDG schemes, Euler scheme and Crank–Nicolson (C—N) scheme, are constructed; In section 3 the stability and error estimate for Euler scheme are derived; In section 4 the theoretical results for Crank–Nicolson scheme are given briefly; Finally, a numerical example is given in sectiom 5.

Throughout context, we shall use letters C, C_i, ε, ε_i to denote some positive constants independent of time-step $\triangle t$ and finite element mesh parameter h, which have different values in different inequalities.

2. Full Discrete Discontinuous Galerkin Schemes

For convenience, let $\bar{\Omega}$ be a polygonal domain, $\mathcal{T}_h = \{k\}$ is a quasi–uniform triangular partition of $\bar{\Omega}$ with mesh parameter $h (0 < h \leq h_0 < 1)$, k is an element in $\mathcal{T}_h$. Let $\triangle t = \tau$ be time–step, $t^n = n\tau$, $n = 0, 1, \cdots, N = [T/\triangle t]$. Suppose that on all time levels $t = t^n (n = 0, 1, \cdots, N)$, the same finite element mesh $\mathcal{T}_h$ for space domain $\bar{\Omega}$ is adopted. Denote

$$V_h = \{v \in L^2(\Omega) : v|_k \in P_r(k), \forall k \in \mathcal{T}_h\}, \tag{2.1}$$

where $P_r(k)$ is a set of polynomials with degree $\leq r$ on k.

I. **Euler FDDG Scheme**

Set $\beta^n(x) = \beta(x, t^n)$. For $\forall k \in \mathcal{T}_h$, let ∂k be the boundary of k which consist of straight line sides l_j $(j = 1, 2, 3)$ and $\gamma(x)$ be the outward unit vector normal to ∂k.