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MIXED FINITE ELEMENT METHODS FOR A STRONGLY NONLINEAR PARABOLIC PROBLEM*[1)]

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Abstract

A mixed finite element method is developed to approximate the solution of a strongly nonlinear second-order parabolic problem. The existence and uniqueness of the approximation are demonstrated and L^2-error estimates are established for both the scalar function and the flux. Results are given for the continuous-time case.

Key words: Finite element method, Nonlinear parabolic problem.

1. Introduction

For second order elliptic problems, the mixed method was described and analyzed by many authors[1−3] in the case of linear equations in divergence form, as well as in [4, 5] for quasilinear or nonlinear problems in divergence form. Johnson and Thomée[6] considered alternative proofs of the previously known error estimates for such methods in the elliptic case. They also analyzed the mixed finite element method for the parabolic equation given by $p_t - \Delta p = f$. Garcia[7] studied the convergence of mixed finite element approximations to quasilinear parabolic equations in the continuous-time case and derived the superconvergent estimates for the difference between the approximate solution and the projection.

In this paper we consider a mixed finite element for approximating the pair (u, p) satisfying second-order, strongly nonlinear parabolic equation

$$\begin{aligned} &u(x,t) = -a(x, \nabla p), \\ &c(x,p)p_t(x,t) + \mathrm{div} u(x,t) = f(x,p,t), \end{aligned} \qquad x \in \Omega, \quad t \in J, \tag{1.1}$$

subject to the following conditions:

$$\begin{aligned} &p(x,0) = p_0(x), \qquad && x \in \Omega, \quad t = 0, \\ &p(x,t) = -g(x,t), \qquad && (x,t) \in \partial\Omega \times J, \end{aligned} \tag{1.2}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded, convex domain with C^2-boundary $\partial\Omega$, and $J = [0, T]$, $a : \overline{\Omega} \times \mathbf{R}^2 \to \mathbf{R}^2$ is cubic continuously differentiable with bounded derivatives through

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third order and has a bounded positive definite Jacobian with respect to the second argument, which implies that ∇p can be locally represented as a function of the flux, say

$$\nabla p = -b(u). \tag{1.3}$$

We shall assume that this representation is global, and that $u \in H^{7/2+\varepsilon_0}(\Omega)^2 \cap C^{0,1}(\overline{\Omega})^2$, $\varepsilon_0 > 0$. Furthermore, assume that the domain of definition of b contains a ball $\mathcal{B}_0$ centered at u in $L^\infty(\Omega)$[5].

The functions $c(x,\nu)$, $f = f(x,\nu,t)$, and $g = g(x,t)$ are continuously differentiable with respect to ν and t. Moreover, there exist constants c_*, c^* and K such that, for all $x \in \overline{\Omega}$, $t \in J$, and $\nu \in \mathbf{R}$,

$$0 < c_* \le c(x,\nu) \le c^*, \tag{1.4}$$

$$|f|, |g|, \left|\frac{\partial c}{\partial \nu}\right|, \ \left|\frac{\partial f}{\partial \nu}\right|, \ \left|\frac{\partial f}{\partial t}\right|, \left|\frac{\partial g}{\partial t}\right| \le K. \tag{1.5}$$

We also assume that the solution $\{u, p\}$ for (1.1)–(1.2) has sufficiently smooth regularity.

2. Formulation of the Mixed Method

Now we let $V = H(\mathrm{div};\Omega) = \{v \in L^2(\Omega)^2\colon \mathrm{div}\, v \in L^2(\Omega)\}$, $W = L^2(\Omega)$. Combining (1.1), (1.2), and (1.3), we arrive at the mixed weak form of (1.1)–(1.2): $(u,p) \in V \times W$ is the solution of the system

$$(b(u), v) - (\mathrm{div} v, p) = \langle g, v\cdot n\rangle, \qquad v \in V, \tag{2.1}$$

$$(c(p)p_t, w) + (\mathrm{div} u, w) = (f(p), w), \qquad w \in W, \tag{2.2}$$

and $p(x,0) = p_0$, where n is the unit exterior normal vector on $\partial\Omega$, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the $L^2(\Omega)$-inner product and the $L^2(\partial\Omega)$-inner product. We consider the Raviart-Thomas[1] space $V_h \times W_h \subset V \times W$ of index $k > 0$ associated with quasiregular partition T_h of Ω by triangles or quadrilaterals, with boundary elements allowed to have one curved side. The mixed finite element method we shall analyzed is the discrete form of (2.1)–(2.2) and is given by: Find $(u_h, p_h) \in V_h \times W_h$ such that $p_h(0) = P(0)$,

$$(b(u_h), v) - (\mathrm{div} v, p_h) = \langle g, v\cdot n\rangle, \qquad v \in V_h, \tag{2.3}$$

$$(c(p_h)p_{ht}, w) + (\mathrm{div} u_h, w) = (f(p_h), w), \qquad w \in W_h, \tag{2.4}$$

where $P(0)$ is the elliptic mixed method projection (to be defined below) into the finite dimensional space W_h of the inital data function p_0.

3. Mixed Method Projection

For introducing an elliptic projection[8], we shall assume that the following boundary value problem

$$\begin{aligned} &-\mathrm{div}(a(\nabla z)) = f(p) - c(p)p_t, \quad \text{in } \Omega,\\ &z = -g, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$