Journal of Computational Mathematics, Vol.17, No.4, 1999, 353-368.

SEMI-DISCRETE AND FULLY DISCRETE PARTIAL PROJECTION FINITE ELEMENT METHODS FOR THE VIBRATING TIMOSHENKO BEAM*1)

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Abstract

In this paper, the partial projection finite element method is applied to the time-dependent problem–the damped vibrating Timoshenko beam model. It is proved that this method allows some new combinations of interpolations for stress and displacement fields. When assuming that a smooth solution exists, we obtain optimal convergence rates with constants independent of the beam thickness.

Key words: Timoshenko beam, Finite element.

1. Introduction

The Timoshenko beam model is given by

$$\begin{cases} -\theta_{xx} + d^{-2}(\theta - \omega_x) = 0 & on \;\; I, \\ d^{-2}(\theta - \omega_x)_x = g(x) & on \;\; I, \\ \theta(0) = \theta(1) = \omega(0) = \omega(1) = 0 \end{cases}$$

where the beam is considered damped, d represents the beam thickness and $I = [0, 1]$. $\theta(x)$ is the rotation of vertical fibers in the beam and $\omega(x)$ is the vertical displacement of the beam's centerline(under a vertical load given by $g(x)$).

Analogous to the situation one would meet in studying the Reissner–Mindlin plate model, the standard finite element methods fail to give good approximation when the beam thickness is too small, owing to a "locking" phenomenon. Instead, mixed methods, based on the introduction of the shear term as a new variable, have proved to be successful([1],[8],etc.). D.N. Arnold[1] studied the discretization with emphasis on the effect of the beam thickness and used a mixed finite element method–reduced integration approach. He obtained optimal-order error estimates with constants independent of the beam thickness.

On the baisis of [11], B.Semper considered the following time-dependent vibrating beam equations

$$\begin{cases} \theta_{tt} + \delta\theta_t - \theta_{xx} + d^{-2}(\theta - \omega_x) = 0 & on \;\; I \times (0, T], \\ \omega_{tt} + \delta\omega_t + d^{-2}(\theta - \omega_x)_x = g(x, t) & on \;\; I \times (0, T], \\ \theta(0) = \theta(1) = \omega(0) = \omega(1) = 0, & \forall t \in [0, T], \\ \theta(x, 0) = \theta_0(x), \theta_t(x, 0) = \theta_1(x), & \forall x \in I, \\ \omega(x, 0) = \omega_0(x), \omega_t(x, 0) = \omega_1(x), & \forall x \in I. \end{cases} \tag{1.1}$$

* Received December 4, 1995.
1) This work was sponsored by State Major Key Project for Basic Research.

Where δ represents a damping constant.

B.Semper discussed some semi-discrete and fully discrete approximations for this model.Following Arnold's idea, he also obtained optimal-order error estimates with constants independent of the beam thickness under the assumption of the regularity of the solution of (1.1), which we will derive in this paper(see Theorem 3.1).

As we have known, in studying of the Reissner–Mindlin plate model, Prof. Zhou Tianxiao [15] proposed a new mixed method:PPM–Partial projection method of finite element discretizations, which has attracted more and more researchers' interest[2,7].In comparison with Galerkin formulations, this method enhanced stability and is promising for the plate and shell problems. In this paper, we extend the idea of PPM to the time-dependent problem. Semi-discrete and fully discrete schemes are proposed for the vibrating beam model (1.1). As desired, this method allows some new combinations of interpolations for stress and displacement fields, and, when assuming a smooth solution, we obtain optimal-order error estimates with constants independent of the beam thickness .

We now give the arrangements of this paper.In section 2 some notations are collected and variational formulations are presented.In section 3 a prioi estimates are derived 3.In the following section new variational formulations are given. In the last two sections semi-discrete approximations and fully discrete approximations are considered and their convergence are analysed .

Throughout this paper we denote by C a constant independent of h and d, which may be different at its each occurrence.

2. Notations and the Original Variational Formulations

At first we introduce some notations. We will use the standard notations for the Sobolev spaces H^r and H_0^r with norm $||\cdot||_r$, with $H^0 = L^2$.The L^2-inner product is denoted by

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Furthermore we denote the dual space of H^{-r} by H^r. For any vectors $\Psi =< \psi_1, \psi_2 >, \Phi =< \phi_1, \phi_2 >\in [H^r]^2$, we interpret

$$(\Psi, \Phi) = (\psi_1, \phi_1) + (\psi_2, \phi_2),$$

$$||\Psi||_r^2 = (\Psi, \Psi)_r \equiv (\psi_1, \psi_1)_r + (\psi_2, \psi_2)_r = ||\psi_1||_r^2 + ||\psi_2||_r^2,$$

(here $(\cdot, \cdot)_r$ represents the $[H^r]^2-$inner product).We also define the following bilinear forms (for abbreviation, we denote $H_0^1(I) = H_0^1, L^2(I) = L^2$ in what follows):
For $< \Psi, \Phi >\in [H_0^1]^2 \times [H_0^1]^2, a(\Psi, \Phi) = ((\psi_1)_x, (\phi_1)_x)$,
For $< \Psi, \eta >\in [H_0^1]^2 \times L^2, b(\eta, \Psi) = (\eta, \psi_1 - (\psi_2)_x)$,
For $< \Psi, \Phi >\in [H_0^1]^2 \times [H_0^1]^2, c(\Psi, \Phi) = (\psi_1 - (\psi_2)_x, \phi_1 - (\phi_2)_x)$,

Given any Banach space V with norm $||\cdot||_V$, for any $v : [0, T] \to V$ which is Lebesgue integrable, we define the norms

$$||v||_{L^p(0,T;V)} = (\int_0^T ||v(\cdot, t)||_V^p dt)^{1/p},\ p = 1, 2$$