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ON THE LEAST SQUARES PROBLEM OF A MATRIX EQUATION*1)

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Abstract

Least squares solution of F=PG with respect to positive semidefinite symmetric P is considered,a new necessary and sufficient condition for solvablity is given,and the expression of solution is derived in the some special cases. Based on the expression, the least spuares solution of an inverse eigenvalue problem for positive semidefinite symmetric matrices is also given.

Key words: Least squares solution, Matrix equation, Inverse eigenvalue problem, Positive semidefinite symmetric matrix.

1. Introduction

The purpose of this paper is to study the least squares problem of the matrix equation F=PG with respect to P$\in S^n_{\geq}$,i.e.

(P_1) $\quad \min\limits_{P\in S^n_{\geq}} \|F-PG\|,\ where\ F,G\in R^{n\times m}\ and\ G\neq 0.$

Where $\|\cdot\|$ denotes the Frobenius norm, and $S^n_{\geq}=\{X\in S^n|X\geq 0\}, S^n=\{X\in R^{n\times n}|X=X^T\}$.Problem$(P_1)$ was first formulated by Allwright [1],A necessary and sufficient condition for the existence of the minimizer $\hat{P}$ in (P_1) was given in [2],where exact global solutions for (P_1) are denoted throughout by $\hat{P}$.The expressions of solution and the numerical solution for (P_1) had been studied in [3]. But the expression of solution is given only for two special cases, i.e. case a): $\hat{P}=FG^+$ if rank(G)=n and $G^TF\in S^m_{\geq}$; and case b):$\hat{P}=0$ iff rank(G)=n and $-FG^T-GF^T\in S^n_{\geq}$.

Problem (P_1) is often appeared in many fields such as structural analysis, system parameter identification ,automatic control, nonlinear programming and so on. A relevant work is [4].

When $S=S^n_{\geq}$,the following inverse eigenvalue problem

$$(P_2')\qquad \min_{P\in S}\|G\wedge -PG\|, where\ G\in R^{n\times m}\ and\ \wedge=diag(\lambda_1,\lambda_2,...,\lambda_m)$$

is a special case of (P_1).A necessary and sufficient condition for solvability and the expression of solution of (P_2') were given for $S=R^{n\times n}$ and $S=S^n$ in [5,6].The

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following special inverse eigenvalue problem

$$(P_2) \qquad \min_{P\in S^n_{\geq}} \|G\wedge -PG\|, where\ G\in R^{n\times m}\ and\ \wedge = diag(\lambda_1,\lambda_2,...,\lambda_m)\geq 0$$

is solved by using dual cone theory[10].

Although the least squares solution of the following problem

$$(P_2'') \qquad \min_{X\in S^n_{\geq}} \|A^TXA-D\|,\ where\ A\in R^{n\times m}, D\in R^{m\times m}$$

was successfully solved by Dai and Lancaster[7], the approach adopted there is based on symmetry of (P_2''), and yet there is not such property in (P_1).So the approach adopted in [7] is not suitable to (P_1).

The aim of this paper is to give a new necessary and sufficient condition for solvability of (P_1) and then derive a expression of solution in the some special cases. Based on the expression we have also solved (P_2).This paper extends the results in [10].

The notation used in the sequel can be summarized as follows. For $A,B\in R^{n\times m}$, A^+ and $A*B$ respectively denote the Moore-Penrose pseudoinverse of A and the Hadamard product of A and B.$OR^{n\times n}$ denotes the set of all orthogonal matrices in $R^{n\times n}$.The notation $A\geq 0(>0)$ means that A is positive semidefinite (definite). For $\Sigma = diag(\sigma_1,\sigma_2,...,\sigma_r)>0, \Phi_\Sigma$ denotes the matrix $(\varphi_{ij})_{r\ \times r}$,where $\varphi_{ij}=(\sigma_i^2+\sigma_j^2)^{-1}, 1\leq i,j\leq r$. In addition, a unit matrix is denoted by I,and the set $\{X\in S^n|X>0\}$ is denoted by $S^n_>$.

This paper is organized as follows. A new necessary and sufficient condition for solvability of (P_1) is given in section 2. Based on the condition, in section 3 the expression of solution of (P_1) is given in some special cases. Problem (P_2) is solved in section 4.

2. The Solvability Conditions for (P_1)

To Study the solvability of (P_1), we decompose the given matrix G by the singular value decomposition(SVD):

$$G = U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T = U_1\Sigma V_1^T \tag{2.1}$$

where $U=(U_1,U_2)\in OR^{n\times n}, U_1\in R^{n\times r}, V=(V_1,V_2)\in OR^{m\times m}, V_1\in R^{m\times r}, \Sigma = diag(\sigma_1,\sigma_2,...,\sigma_r)>0, r=rank(G)$.

Theorem 2.1. *Suppose that rank(G)<n, and a SVD of the matrix G is (2.1). Then (P_1) has a solution if and only if rank($\hat{P}_{11}$)= rank($\hat{P}_{11}|\hat{P}_{12}$),where $\hat{P}_{11}$ is a unique minimizer of $\|U_1^TFV_1-P_{11}\Sigma\|$ with respect to $P_{11}\in S^r_{\geq}$,and $\hat{P}_{12}=(U_2^TFV_1\Sigma^{-1})^T$.*

If (P_1) has a solution, then the expression of solution is

$$\hat{P} = U\begin{pmatrix}\hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{12}^T & \hat{P}_{12}^T\hat{P}_{11}^+\hat{P}_{12}+B\end{pmatrix}U^T \tag{2.2}$$

where $B\in S^{n-r}_{\geq}$ is arbitrary.

To prove Theorem 2.1, it will be convenient to give the following three lemmas.

Lemma 2.1.[1] *The minimizer in (P_1) exists and is unique when rank(G)=n.*