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SOME SUPERCONVERGENCE RESULTS OF WILSON-LIKE ELEMENTS*

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Abstract

In this paper, the superconvergence of a class of Wilson-like elements is considered. A superconvergent estimate on the centers of elements and some superconvergence recoveries on the four vertices and the four midpoints of edges of elements are also obtained for piecewise strongly regular quadrilateral subdivisions.
Key words: Superconvergence, Wilson-like element.

1. Introduction

The superconvergence of finite element methods is an important property both in theory and in practice. Many superconvergence results about conforming finite element methods have been obtained (see [4] [17]). But there are also many nonconforming finite element methods in computational practice, in addition to conforming ones. Do the superconvergence results still hold for those nonconforming finite element methods ? The Wilson element is one of the most important nonconforming finite element. [14] first studied the superconvergence property of Wilson element, and obtained the superconvergent estimate of the gradient error on the centers of elements, on an average sense. [1] and [12] sharpened this result, and obtained the pointwise superconvergent estimate. But, as we know, Wilson element may cause divergence for the arbitrary quadrilateral meshes (see [5] [10]), and the results obtained in [14], [1] and [12] only hold for rectangular meshes, and only for the partial differential equations which do not involve mixed derivative terms. [3] and [11] presented a class of so-called Wilson-like nonconforming elements, which converge for arbitrary quadrilateral meshes. [15] studied the superconvergence property of those Wilson-like elements, and obtained a superconvergent estimate of gradient error on the centers of elements, on the average sense. In this paper, we study the pointwise superconvergence property of those

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Wilson-like elements, and prove that Wilson-like elements, if the nonconforming basis functions are even polynomials, i.e., $b_i = 0$ $(1 \le i \le n)$ (see Theorem 4.1), have the superconvergence at the centers of elements, at the four vertices and the midpoints of four edges of elements, provided that the quadrilateral subdivision is piecewise strongly regular. In the last section, some numerical examples are presented to illustrate the theoretical results and the necessity of the condition $b_i = 0$ $(1 \le i \le n)$.

2. Basic Notions

Consider the following Dirichlet problem of a second order elliptic equation,

$$\begin{cases} -\sum_{i,j=1}^{2} D_j(\alpha_{ij} D_i u) + \beta u = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a convex polygonal domain, $\partial\Omega$ is the boundary of Ω, $f \in L^2(\Omega)$, functions α_{ij}, β are sufficiently smooth, α_{ij} satisfy the ellipticity condition and $\beta \ge 0$. The partial differential operators $D_j (j = 1, 2)$ mean $D_1 = \frac{\partial}{\partial x}$, $D_2 = \frac{\partial}{\partial y}$ respectively.

The variational problem of the equation (2.1) is: Find $u \in H_0^1(\Omega)$ so that

$$a(u, v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u, v) = \int_\Omega \Big(\sum_{i,j=1}^{2} \alpha_{ij} D_i u D_j v + \beta u v \Big) dxdy, \quad (f, v) = \int_\Omega f v dxdy.$$

Applying the definition of strongly regular subdivision (see Lesaint and Zlámal [6], Zhu and Lin [17], and Zlámal [18]), similar to the definition of piecewise strongly regular triangulation (see Lin and Lü [7], Lin and Xu [8]), we define the piecewise strongly regular quadrilateral subdivision by (also see Zhang and Li [16])

Definition. *A quadrilateral subdivision T_h on Ω is called piecewise strongly regular subdivision, if Ω is divided into finite quadrilateral subdomains Ω_i $(1 \le i \le N)$ without inner vertices, and the subdivision restricted on each Ω_i is strongly regular.*

$\forall K \in T_h$, let $A_i(x_i, y_i) (1 \le i \le 4)$ denote the four vertices of the element K, $A(x_0, y_0)$ be the center. h_K means the diameter of K, i.e., $h_K = \operatorname{diam}(K)$, $h = \max_{K \in T_h} h_K$. In this paper, C means a generic constant independent of K and h, and may have different values at different places. The notations of Sobolev spaces and their norms used in this paper are as the same as those in Ciarlet [2].

Let $\widehat{K} = [-1, 1] \times [-1, 1]$ be the reference square having the vertices $\widehat{A}_i (1 \le i \le 4)$. For every element K, there exists a unique one-to-one mapping $F_K : \widehat{K} \longrightarrow K$, given by $x = \sum_{i=1}^{4} x_i \widehat{N}_i(\xi, \eta)$, $y = \sum_{i=1}^{4} y_i \widehat{N}_i(\xi, \eta)$, where $\widehat{N}_i(\xi, \eta) = (1 \pm \xi)(1 \pm \eta)/4$ $(1 \le i \le 4)$ are the bilinear shape functions on $\widehat{K}$.