Journal of Computational Mathematics, Vol.16, No.3, 1998, 239–256.

THE APPROXIMATIONS OF THE EXACT BOUNDARY CONDITION AT AN ARTIFICIAL BOUNDARY FOR LINEARIZED INCOMPRESSIBLE VISCOUS FLOWS*[1)]

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Abstract

We consider the linearized incompressible Navier-Stokes (Oseen) equations in a flat channel. A sequence of approximations to the exact boundary condition at an artificial boundary is derived. Then the original problem is reduced to a boundary value problem in a bounded domain, which is well-posed. A finite element approximation on the bounded domain is given, furthermore the error estimate of the finite element approximation is obtained. Numerical example shows that our artificial boundary conditions are very effective.

Key words: Oseen equations, Artificial boundary, Artificial boundary condition, Finite element approximation, Error estimate.

1. Introduction

Many problems arising in fluid mechanics are given in an unbounded domain, such as fluid flow around obstacles. When computing the numerical solutions of these problems, one often introduces artificial boundaries and sets up artificial boundary conditions on them. Then the original problem is reduced to a problem in a bounded computational domain. In order to limit the computational cost these boundaries must be not too far from the domain of interest. Therefore, the artificial boundary conditions must be good approximate to the "exact" boundary conditions (i.e. such that the solution of the problem in the bounded domain is equal to the solution of the original problem). Thus the accuracy of the artificial boundary conditions and the computational cost are closely related. It has often been studied during the last ten years to design artificial boundary conditions with high accuracy on a given artificial boundary for solving partial differential equations on an unbounded domain. For example, Goldstein[5], Feng[4], Han and Wu[14,15], Hagstrom and Keller[6,7], Halpern[8], Halpern and Schatzman[9], Nataf[17], Han, Lu and Bao[13], Han and Bao[11,12], Bao[1] and others have studied how to design the artificial boundary conditions for solving partial differential equations in an unbounded domain.

* Received July 3, 1995.
1) This work was supported by the Climbing Program of National Key Project of Foundation and Doctoral Program foundation of Institution of Higher Education. Computation was supported by the State Key Lab. of Scientific and Engineering Computing.

In this paper we consider the linearized incompressible Navier-Stokes equations in a slip flat channel. It is an approximate problem of two-dimensional steady incompressible viscous folw around obstacles. We derived a solution which can be written in the form of Fourier series in the unbounded domain by the method of separation of variables. Then the exact and a series of approximate artificial boundary conditions are derived by the continuity of velocity and the normal stress at the artificial boundary. Therefore the original problem is reduced to a series of problems in a bounded computational domain. Particularly, a finite element approximation on the bounded domain is given, and the error estimate of the finite element approximation is obtained. Numerical example shows the effectiveness of the artificial boundary condition.

2. Oseen Equations and their Solution

Let Ω_i be an obstruction in a channel defined by $\mathbb{R} \times (0, L)$ and $\Omega = \mathbb{R} \times (0, L) \setminus \bar{\Omega}_i$. Consider the following Oseen equations:

$$a\frac{\partial u}{\partial x_1} + \nabla p = \nu \triangle u, \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{2.2}$$

with boundary conditions

$$u_2|_{x_2=0,L} = 0,\ \sigma_{12}|_{x_2=0,L} = \nu\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big)\Big|_{x_2=0,L} = 0,\ -\infty < x_1 < +\infty, \tag{2.3}$$

$$u|_{\partial\Omega_i} = 0, \tag{2.4}$$

$$u(x) \to u_\infty = (a, 0)^T, \quad \text{when} \quad x_1 \to \pm\infty; \tag{2.5}$$

where $u = (u_1, u_2)^T$ is the velocity, p is the pressure, $\nu > 0$ is the kinematic viscosity, $x = (x_1, x_2)^T$ is coordinate, $a > 0$ is a constant and σ_{12} is the tangential stress on the wall. Obviously condition (2.3) is equivalent to the following condition:

$$\frac{\partial u_1}{\partial x_2}\Big|_{x_2=0,L} = u_2|_{x_2=0,L} = 0. \tag{2.6}$$

Taking two constants $b < d$, such that $\Omega_i \subset (b, d) \times (0, L)$, then Ω is divided into three parts Ω_b, Ω_T and Ω_d by the artificial boundaries Γ_b and Γ_d with

$$\begin{aligned}
&\Gamma_b = \{x \in \mathbb{R}^2 |\ x_1 = b,\ 0 \le x_2 \le L\},\\
&\Gamma_d = \{x \in \mathbb{R}^2 |\ x_1 = d,\ 0 \le x_2 \le L\},\\
&\Omega_b = \{x \in \mathbb{R}^2 |\ -\infty < x_1 < b,\ 0 < x_2 < L\},\\
&\Omega_T = \{x \in \mathbb{R}^2 |\ b < x_1 < d,\ 0 < x_2 < L\} \setminus \bar{\Omega}_i,\\
&\Omega_d = \{x \in \mathbb{R}^2 |\ d < x_1 < +\infty,\ 0 < x_2 < L\}.
\end{aligned}$$

We now consider the Oseen equations on the unbounded domain Ω_d:

$$a\frac{\partial u}{\partial x_1} + \nabla p = \nu \triangle u, \quad \text{in } \Omega_d, \tag{2.7}$$