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NOTES ON REFINABLE FUNCTIONS*1)

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Abstract

In this paper some properties of refinable functions and some relationships between the mask symbol and the refinable functions are studied. Especially, it is illustrated by examples that the linear spaces formed by the translates over the lattice points of refinable functions may contain polynomial spaces of degree higher than the smooth order of the corresponding refinable functions.

Key words: Mask, Symbol, Refinable Function.

1. Introduction

It is well-known that refinable functions play an important role in the studying of wavelet. Usually, one hopes that refinable functions have some particular properties such as smoothness and integrability. In this note, the zeros of an integrable refinable function are obtained. In particalar by examples one shows that the linear space associating the translates over the lattice points of a refinable function could include polynomial space of degree higher than its smooth order.

Let s be a positive integer and let $\mathbf{R}^s$ (resp. $\mathbf{C}^s$) be the s-dimensional real (complex) space equipped with the norm $|\cdot|$ given by

$$|x| = \Big(\sum_{j=1}^{s} |x_j|^2\Big)^{\frac{1}{2}} \quad \text{for} \quad x = (x_1, \cdots, x_s) \in \mathbf{R}^s \quad (\text{resp.} \quad \mathbf{C}^s).$$

By a *mask* $\mathbf{a} = \{a_\alpha;\ \alpha \in \mathbf{Z}^s\}$ we mean a mapping of finitely supported from $\mathbf{Z}^s$ to $\mathbf{C}$. For $1 \le p \le \infty$, we use $L_p = L_p(\mathbf{R}^s)$ to denote the Banach space of all functions f on $\mathbf{R}^s$ such that

$$\|f\|_p := \Big(\int_{\mathbf{R}^s} |f(x)|^p\Big)^{\frac{1}{p}} < \infty,$$

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where f could be complex valued.

A function on $\mathbf{R}^s$ is called refinable if it satisfies

$$\varphi(x) = \sum_{\alpha \in \mathbf{Z}^s} a_\alpha \varphi(2x - \alpha), \quad x \in \mathbf{R}^s \tag{1}$$

for a mask $\mathbf{a} = \{a_\alpha, \alpha \in \mathbf{Z}^s\}$, and

$$p(z) = 2^{-s} \sum_{\alpha \in \mathbf{Z}^s} a_\alpha z^\alpha \tag{2}$$

is called the *symbol* of the mask $\mathbf{a}$.

$\varphi \in \mathcal{L}^p$ is called L_p-stable if there exists a positive constant c of independent with $\mathbf{a}$ and φ such that

$$\|\mathbf{a}\|_p \le c \Big\| \sum_{\alpha \in Z^s} a_\alpha \varphi(x - \alpha) \Big\|_p,$$

where by $\varphi \in \mathcal{L}^p$ we mean

$$\varphi^0 := \sum_{\alpha \in \mathbf{Z}^s} |\varphi(x - \alpha)| \in L_p([0,1]^s)$$

(see [7]) and by $l_p := l_p(\mathbf{Z}^s)$ we denote the Banach space of all elements a defined on $\mathbf{Z}^s$ for which

$$\|a\|_p = \Big(\sum_{\alpha \in \mathbf{Z}^s} |a_\alpha|^p \Big)^{\frac{1}{p}} < \infty$$

equipped with the norm $\|\cdot\|$.

Note 1: Let φ be a continuous function with compact support. Then the (see [2] Theorem 4.1 and Theorem 4.2) the L_p-stability of φ for some $1 \le p \le \infty$ implies that of φ for all $1 \le p \le \infty$. Furthermore, it is easy to show that L_p-stability of the function φ is equivalent to the l_∞ linear independence, where by $\{\varphi(x - \alpha), \alpha \in \mathbf{Z}^s\}$ being l_p linearly dependent we mean that there exists $0 \ne \lambda \in l_p$ such that

$$\sum_{\alpha \in \mathbf{Z}^s} \lambda_\alpha \varphi(x - \alpha) \equiv 0, \ x \in \mathbf{R}^s.$$

In fact, if $\{\varphi(x - \alpha), \alpha \in \mathbf{Z}^s\}$ are l_∞ linearly independent, then φ is L_∞-stable (see [2] page 24). In other way round, it is easy to see that $\{\varphi(x - \alpha), \alpha \in \mathbf{Z}^s\}$ being l_∞ linearly dependent implies $\sum_{\alpha \in \mathbf{Z}^s} \lambda_\alpha \varphi(x - \alpha) \equiv 0$ for some $0 \ne \lambda \in l_\infty$. Therefore, φ could not be L_∞-stable.

Finally, in this note we will use $\hat{f}(u) = \int_{\mathbf{R}^s} f(x) e^{-iu \cdot x} dx$ $(u \in \mathbf{C}^s)$ to denote the Fourier-Laplace transform of f, where for $u = (u_1, \cdots, u_s) \in \mathbf{C}^s$ and $x = (x_1, \cdots, x_s) \in \mathbf{R}^s$, $u \cdot x = \sum_{j=1}^s u_j x_j$. Restricted to $\mathbf{R}^s$, $\hat{f}$ become the Fourier transform of f.

2. Main Results

Our first result will be about the Fourier-Laplace transform of a refinable function. From [2], we know that if $\varphi \in L_1$ is refinable, then $\hat{\varphi}$ is an entire function and