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ON MAXIMUM NORM ESTIMATES FOR RITZ-VOLTERRA PROJECTION WITH APPLICATIONS TO SOME TIME DEPENDENT PROBLEMS*1)

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Abstract

The stability in L^∞-norm is considered for the Ritz-Volterra projection and some applications are presented in this paper. As a result, point-wise error estimates are established for the finite element approximation for the parabolic integro-differential equation, Sobolev equations, and a diffusion equation with non-local boundary value problem.

1. Introduction

We are concerned with the finite element method for parabolic integro-differential equation

$$\begin{aligned} &u_t(t) + V(t)u(t) = f(t), \qquad t \in (0, T), \\ &u(0) = v, \end{aligned} \tag{1.1}$$

where $V(t)$ is in general an integro-differential operator defined on a Hilbert space X and that u and f are X-valued functions defined on $J = (0, T)$ with a positive time T. A typical example of the Hilbert space X in the application will be the Sobolev space H_0^1 consisting of functions defined on an open bounded domain Ω with vanished boundary value and first order derivatives summable in L^2, while the operator $V(t)$ is the one defined by

$$V(t)u(t) \equiv A(t)u(t) + \int_0^t B(t,\tau)u(\tau)d\tau, \quad \text{in } \Omega \tag{1.2}$$

for any $u(t) \in H_0^1(\Omega)$, where $A(t)$ is a linear elliptic operator of second order and that $B(t, \tau)$ any linear operator of no more than second order. Although more examples of integro-differential operators will be considered in this paper, we shall illustrate our results for the operator $V(t)$ defined by (1.2), since the others can be modified to fit the strategy designed for (1.2).

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Numerical methods to the equation (1.1) have been studied by several authors recently. For finite difference schemes we refer to [23] and the references cited therein. The finite element method for this problem has also been studied; in [23] both smooth and non-smooth data cases were considered and optimal error estimates in L^2 were obtained, the semi-linear equation with non-smooth data and an operator B of zero order was treated in [12] along with a particular attention paid to the computation of the memory term by the quadrature rule. Recently, a different approach to the error analysis was proposed in [3] and [4]. Their idea can be summerized as introducing a so-called Ritz-Volterra projection to decompose the error. A systematic study of Ritz-Volterra projection and its applications to parabolic and hyperbolic integro-differential equations, Sobolev equation, and the equations of visco-elasticity can be found from [14].

For the sake of convenience of the analysis, we shall take Ω to be a plane convex polygonal domain. Let $\mathcal{T}_h$ be a quasi-uniform triangulation so that $\Omega_h = \cup_{K\in\mathcal{T}_h} K = \Omega$. Let S_h be the finite element subspace associated with $\mathcal{T}_h$. Without loss of generality, we shall assume that S_h is made up of piece-wise linear functions.

The object of this paper is to study the convergence behavior of the finite element approximation in the L^∞-norm. As a matter of fact, this problem had been considered by Lin and Zhang[15], where an optimal maximum norm has been obtained for piecewise linear elements for a very special case, that is, the operators A and B are divergence form which allows us to use the standard regularized Green function[16,22], and by Lin, Thomee, and Wahlbin in [14], where the following estimate for any small $\varepsilon > 0$

$$\|u(t) - u_h(t)\|_{0,\infty} \le C(u, \varepsilon)h^{r-\varepsilon}$$

was derived based on their estimate in L^p. Here r is the optimal order in the approximation and $C(u)$ a constant dependent upon the exact solution u only. It is clear that such an estimate is not optimal in compare with the results for the elliptic and parabolic equations[17,22,10,20,18]. We shall, therefore, study this problem from a different point of view in order to get a sharp estimate in the L^∞-norm. The main idea of our approach can be summarized as firstly introducing an auxiliary problem associated with the Ritz-Volterra operator V and then establishing our main results with the help of the solution of this auxiliary problem. The auxiliary problem to be introduced in next section is an analogy of the regularized Green's function in the study of the L^∞-stability for the elliptic equation of second order. Thus, the only contribution of the authors would be to apply the known technique appropriately to the current problem. However, such an extension is not trivial due to the memory term involved in the operator V.

Our main result regards to the maximum norm error estimate for the Ritz-Volterra projection V_h defined by

$$V(t; V_h u(t), \phi) = V(t; u(t), \phi), \quad \phi \in S_h \tag{1.3}$$

for each $t \in J$, where $V(t; \cdot, \cdot)$ is the bilinear form associated with the Ritz-Volterra operator $V(t)$ defined by

$$V(t; u(t), v(t)) = A(t; u(t), v(t)) + \int_0^t B(t, \tau; u(\tau), v(t))d\tau \tag{1.4}$$