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CONSTRUCTION OF VOLUME-PRESERVING DIFFERENCE SCHEMES FOR SOURCE-FREE SYSTEMS VIA GENERETING FUNCTIONS*1)

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1. Introduction

Source-free dynamical systems are of great importance in many branches of physics. A significant subject for such systems is to design "proper" numerical algorithms. Since the phase flows of source-free systems are volume-preserving transformations on the corresponding phase space, the proper way consists in the requirement that the step-transition maps of the algorithms are volume-preserving. We call such algorithms volume-preserving algorithms. In [5], Thyagaraja and Haas designed volume-preserving algorithms for 3-dimensional source-free systems based on a type of generating function representations of volume-preserving mappings on R^3. in [1], Feng Kang and the author gave a more general method to construct volume-preserving difference schemes for general n-dimensional source-free systems based on the decomposition of a source-free vector field on R^n into a sum of $n-1$ essentially 2-dimensional Hamiltonian vector fields and on the well known symplectic difference schemes for 2-dimensional Hamiltonian systems. In this paper, we present another general method for the same purpose whose basis is the generating function apparatus for volume-preserving mappings and Hamilton-Jacobi theory for source-free systems, which have both been well developed by the author[4]. We emphasize that our method presented in this paper provides volume-preserving algorithms for arbitrarily dimensional source-free systems with arbitrarily high order of accuracy which are implicit only in one coordinate and therefore, is superior to the methods given in [5] and in [1] which only provide first order and highly implicit volume-preserving difference schemes respectively.

2. Basic Theorems

The theorems in this section are important to the construction of volume-preserving difference schemes for source-free systems in the next section.

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Theorem 2.1. *Let* $\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix} \in GL(2n)$. *Denote* $\alpha^{-1} = \begin{pmatrix} A^\alpha & B^\alpha \\ C^\alpha & D^\alpha \end{pmatrix}$. *Let* $g : R^n \to R^n$ *be a differentiable volume-preserving mapping satisfying the transversality condition*

$$\left| C_\alpha \frac{\partial g}{\partial z}(z) + D_\alpha \right| \neq 0 \tag{2.1}$$

in a neighborhood of some $z_0 \in R^n$. *Then there exists a differentiable mapping* $f(w) = f_{\alpha,g} = (f_1(w), f_2(w), \cdots, f_n(w))^T$ *satisfying condition*

$$\left| \frac{\partial f}{\partial w}(w) C_\alpha - A_\alpha \right| = \left| B_\alpha - \frac{\partial f}{\partial w}(w) D_\alpha \right| \neq 0 \tag{2.2}$$

in a neighborhood of the point $w_0 = C_\alpha g(z_0) + D_\alpha z_0$ *in* R^n *such that the mapping* $\hat{z} = g(z)$ *can be reconstructed from* $f = f_{\alpha,g}$ *by the relation*

$$A_\alpha \hat{z} + B_\alpha z = f(C_\alpha \hat{z} + D_\alpha) \tag{2.3}$$

in a neighborhood of the point z_0 *in* R^n. *Conversely, let* $f(w) = (f_1(w), \cdots, f_n(w))^T$ *be a differentiable mapping satisfying condition* (2.2) *in a neighborhood* W *of the point* w_0 *in* R^n. *Then the relation* (2.3) *gives a volume-preserving mapping* $\hat{z} = g(z)$ *satisfying the transversality condition* (2.1) *in a neighborhood of the point* $z_0 = C^\alpha f(w_0) + D^\alpha w_0$ *in* R^n.

Remark 1. Locally speaking, a volume-preserving mapping is completely given from matrix $\alpha \in GL(2n)$ and mapping $f = f_{\alpha,g}$ by the relation (2.3). We call $f = f_{\alpha,g}$ the generating mapping of the type α and the mapping g.

Remark 2. Matrix α represents the type of generating mappings. Specifically we consider some important cases of α. For example, we take

$$\alpha_{(s,s)} = \begin{pmatrix} I_n - E_{ss} & E_{ss} \\ E_{ss} & I_n - E_{ss} \end{pmatrix}, \quad 1 \leq s \leq n, \tag{2.4}$$

where E_{ss} denotes an $n \times n$ matrix of which only entry at the s-th row and s-th column is 1 and all other entries are 0. In this case, equations (2.2) and (2.3) have much more simple forms. For $\alpha = \alpha_{(1,1)}$, for example, (2.2) turns into

$$\frac{\partial f_1}{\partial w_1} = \left| \frac{\partial (f_2, \cdots, f_n)}{\partial (w_2, \cdots, w_n)} \right| \neq 0 \tag{2.5}$$

and (2.3) turns into

$$\begin{cases} z_1 = f_1(\hat{z}_1, z_2, \cdots, z_n), \\ \hat{z}_2 = f_2(\hat{z}_1, z_2, \cdots, z_n), \\ \quad \vdots \\ \hat{z}_n = f_n(\hat{z}_1, z_2, \cdots, z_n). \end{cases} \tag{2.6}$$

For $\alpha_{(s,s)}$, the results are similar.

Remark 3. We note that from (2.5), among the n components of the generating mapping $f(w) = (f_1(w), f_2(w), \cdots, f_n(w))^T$ of the type $\alpha_{(1,1)}$, the last $n-1$ components