

## SPLINE FINITE DIFFERENCE METHODS AND THEIR EXTRAPOLATION FOR SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS\*

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### Abstract

In this paper we consider a class of singular two-point boundary value problem:  $(x^\alpha u')' = f(x, u)$ ,  $u(0) = A$ ,  $u(1) = B$ ,  $0 < \alpha < 1$ . The asymptotic expression of the spline finite difference solution for the problem is obtained. By using the asymptotic expression, Richardson's extrapolation can be done and the accuracy of numerical solution can be improved. Some numerical examples are given in illustration of this theory.

### §1. Introduction

Consider the singular two-point boundary value problem:

$$\begin{cases} (x^\alpha u')' = f(x, u), & 0 < x \leq 1, \\ u(0) = A, & u(1) = B. \end{cases} \quad (1)$$

Here  $A$  and  $B$  are finite constants, and  $\alpha \in (0, 1)$  which may also take 1 or 2. We assume:

(A) for  $(x, u) \in \{[0, 1] \times R\}$ ,  $f(x, u)$  is continuous,  $\partial f / \partial u$  exists and is continuous, and  $\partial f / \partial u \geq 0$ .

It is well known that (1) has a unique solution. Problem (1) has been extensively discussed. In the linear case, Jamet [1] considered a standard three-point finite difference scheme with uniform mesh and has shown that the error is  $O(h^{1-\alpha})$  in maximum norm. Ciarlet [2] considered the application of Rayleigh-Ritz-Galerkin method and improved Jamet's result by showing that the error is  $O(h^{2-\alpha})$  in uniform mesh for their Galerkin approximation. Gustafsson [3] gave a numerical method for linear problems by representing the solutions as series expansions on a subinterval near the singularity and using the difference method for the remaining interval. He constructed a compact second order, fourth order scheme. Reddien [4] and Reddien and Schumaker [5] have considered the collocation method for singular two-point boundary value problems and studied the existence, uniqueness and convergence rates of these methods. Chawla

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and Katti [6] have constructed three kinds of difference methods and shown that these methods are  $O(h^2)$ -convergent. Recently, Iyengar and Jain [7] have discussed the construction of a spline function for a class of singular two-point boundary value problem:  $x^{-\alpha}(x^\alpha u')' = f(x, u), u(0) = A, u(1) = B, 0 < \alpha < 1$  or  $\alpha = 1, 2$ . The boundary conditions may also be of form  $u'(0) = 0, u(1) = B$ . They showed that the spline finite difference methods are  $O(h^2)$ -convergent when  $0 < \alpha < 1, \alpha = 2$ . Han<sup>[8]</sup> has shown, for  $\alpha = 1$ , that the spline finite difference method is also  $O(h^2)$ -convergent.

In this paper, first we give the spline finite difference scheme for the singular two-point boundary value problem which is similarly constructed in [7]. Secondly, we show that the asymptotic expression of the spline finite difference solution exists. So Richardson's extrapolation can be done. This will greatly increase the accuracy of numerical solution. Finally, some numerical examples are given in illustration of this theory.

### §2. The Spline Finite Difference Method

Let  $\Delta$  be an equidistant partition of  $[0, 1]$ :

$$\Delta : 0 = x_0 < x_1 < \dots < x_N = 1.$$

Set  $\tilde{u}(x_j) = \tilde{u}_j$  and  $f(x_j, \tilde{u}_j) = f_j$ . We write

$$(x^\alpha \tilde{u}')' = \frac{f_{j-1}}{h}(x_j - x) + \frac{f_j}{h}(x - x_{j-1}) \quad x_{j-1} < x < x_j \tag{2}$$

where  $h = x_{j+1} - x_j = 1/N$ . Integrating (2), dividing by  $x^\alpha$ , and then integrating we obtain

$$\tilde{u}(x) = \frac{f_{j-1}}{h} \left( \frac{-x^{3-\alpha}}{2(3-\alpha)} + \frac{x_j x^{2-\alpha}}{2-\alpha} \right) + \frac{f_j}{h} \left( \frac{x^{3-\alpha}}{2(3-\alpha)} - \frac{x_{j-1} x^{2-\alpha}}{2-\alpha} \right) + c_j x^{1-\alpha} + D_j$$

where  $c_j$  and  $D_j$  can be determined by interpolation conditions  $\tilde{u}(x_{j-1}) = \tilde{u}_{j-1}, \tilde{u}(x_j) = \tilde{u}_j$ . We get the spline approximation as

$$\begin{aligned} \tilde{u}(x) = & \frac{1}{S_j} (x_j^{1-\alpha} \tilde{u}_{j-1} - x_{j-1}^{1-\alpha} \tilde{u}_j) + \frac{1}{S_j} (\tilde{u}_j - \tilde{u}_{j-1}) x^{1-\alpha} + \frac{f_{j-1}}{h} \left\{ -\frac{x^{3-\alpha}}{2(3-\alpha)} + \frac{x_j x^{2-\alpha}}{2-\alpha} \right. \\ & - \frac{1}{S_j} \left[ \frac{1}{2-\alpha} x_j (x_j^{2-\alpha} - x_{j-1}^{2-\alpha}) - \frac{1}{2(3-\alpha)} (x_j^{3-\alpha} - x_{j-1}^{3-\alpha}) \right] x^{1-\alpha} \\ & + \left. \frac{1}{S_j} \left[ \frac{h}{2-\alpha} x_j^{2-\alpha} x_{j-1}^{1-\alpha} - \frac{h}{2(3-\alpha)} x_j^{1-\alpha} x_{j-1}^{1-\alpha} (x_j + x_{j-1}) \right] \right\} \\ & + \frac{f_j}{h} \left\{ \frac{x^{3-\alpha}}{2(3-\alpha)} - \frac{x_{j-1} x^{2-\alpha}}{2-\alpha} + \frac{1}{S_j} \left[ \frac{1}{2-\alpha} x_{j-1} (x_j^{2-\alpha} - x_{j-1}^{2-\alpha}) \right. \right. \\ & - \frac{1}{2(3-\alpha)} (x_j^{3-\alpha} - x_{j-1}^{3-\alpha}) \left. \right] x^{1-\alpha} - \frac{1}{S_j} \left[ \frac{h}{2-\alpha} x_j^{1-\alpha} x_{j-1}^{2-\alpha} \right. \\ & \left. \left. - \frac{h}{2(3-\alpha)} x_j^{1-\alpha} x_{j-1}^{1-\alpha} (x_j + x_{j-1}) \right] \right\}, \quad x_{j-1} < x < x_j \tag{3} \end{aligned}$$