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APPROXIMATE METHODS FOR GENERALIZED INVERSES OF OPERATORS IN BANACH SPACES*

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Abstract

In this paper, we present the necessary and sufficient condition of convergence of several iterative methods for computing the generalized inverses of operators in Banach spaces. It is proved that the iterative methods converge to the generalized inverse of an Operator in Banach spaces if and only if these conditions are satisfied.

1. Introduction

In this paper, we will present the necessary and sufficient condition of convergence of the iterative methods for computing the generalized inverses of operators. Let X and Y be two Banach spaces, and $B[X, Y]$ be the Banach space consisting of all bounded linear operators from X into Y. $D(T)$, $R(T)$ and $N(T)$ denote the domain, range and null of T respectively. We assume that the closed subspace $N(T)$ of x has a topological complement $N(T)^c$ and the closed subspace $\overline{R(T)}$ of Y has a topological complement $\overline{R(T)}^c$. Thus,

$$X = N(T) \bigoplus N(T)^c, \quad Y = \overline{R(T)} \bigoplus \overline{R(T)}^c.$$

The above decomposition exists if and only if there exist projectors P and Q such that

$$PX = N(T), \quad QY = \overline{R(T)}.$$

In this case, Nashed [1] pointed out that the operator T has unique generalized inverses $T^+ \equiv T^+_{P,Q}$ ($T^+_{P,Q}$ implies that the operator T^+ depends on the projectors P and Q) such that

$$\begin{cases} D(T^+) = R(T) \bigoplus \overline{R(T)}^c, \quad N(T^+) = \overline{R(T)}^c, \\ R(T^+) = N(T)^c, \quad TT^+T = T, \quad T^+TT^+ = T^+, \quad \text{on} \quad D(T^+), \\ T^+T = I - P, \quad TT^+ = Q|_{D(T^+)}, \end{cases} \tag{1}$$

where $Q|_{D(T+)}$ is the restriction of Q on $D(T^+)$. In addition, T^+ is bounded if and only if $R(T)$ is closed in Y. In this paper, $R(T)$ is assumed to be closed. Then, we

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obviously have

$$\begin{cases} X = N(T) \bigoplus N(T)^c, \quad Y = R(T) \bigoplus R(T)^c, \\ D(T^+) = Y, \quad N(T^+) = R(T^c, \quad R(T^+) = N(T)^c \end{cases} \tag{2}$$

and

$$\begin{cases} TT^+T = T, \quad T^+TT^+ = T^+, \\ T^+T = P_{N(T)^C}, \quad TT^+ = P_{R(T)}. \end{cases} \tag{3}$$

From (3) we can easily obtain

$$\begin{cases} T^+P_{R(T)} = T^+, \quad P_{N(T)^C}T^+ = T^+, \\ TP_{N(T)^C} = T, \quad P_{R(T)}T = T. \end{cases} \tag{4}$$

2. The Newton-Raphson Iterative Methods

Theorem 1. *Let*

$$R_k = P_{R(T)} - TT_k, \quad T_{k+1} = T_k [I + R_k], \quad k \geq 0. \tag{5}$$

The sequence $\{T_k\}$ converges quadratically to T^+ if and only if

$$P_{N(T)^C}T_0 = T_0 \quad \text{and} \quad r_\sigma(R_0) < 1, \tag{6}$$

where I is the identity operator in Y, and $r_\sigma(R_0)$ is the spectral radius of R_0.

Proof. Suppose condition (6) holds. Then we can show that $P_{R(T)}R_k = R_k$ and $P_{N(T)^C}T_k = T_k$, for any $K = 0, 1, 2, ...$ [3]. Thus,

$$\begin{aligned} R_k &= P_{R(T)} - TT_k = P_{R(T)} - TT_{k-1} [I + R_{k-1}] \\ &= R_{k-1} - TT_{k-1}R_{k-1} = P_{R(T)}R_{k-1} - TT_{k-1}R_{k-1} \\ &= R_{k-1}^2 = \cdots = R_0^{2^k} \end{aligned}$$

and

$$T^+R_k = T^+P_{R(T)} - T^+TT_k = T^+ - T_k. \tag{7}$$

Since $r_\sigma(R_0) < 1$, we have

$$\lim_k \|T^+ - T_k\| \leq \lim_k \|T^+\| \cdot \|R_0^{2^k}\| = 0.$$

Inversely, suppose $\|T_k - T^+\| \to 0$ as $K \to \infty$. Then,

$$\|R_0^{2^k}\| = \|TT^+ - TT_k\| \leq \|T\| \cdot \|T^+ - T_k\| \to 0$$

or

$$r_\sigma (R_0)^{2^k} = r_\sigma \left(R_0^{2^k}\right) \leq \|R_0^{2^k}\| \to 0.$$

Thus,

$$r_\sigma (R_0) < 1. \tag{8}$$