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CONVERGENCE AND STABILITY PROPERTIES OF A VECTOR PADÉ EXTRAPOLATION METHOD*

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Abstract

In this paper we introduce a new convergence accelerator for vector sequences — vector Padé approximation method(VPA), discuss its convergence and stability properties, and show that it is a bona fide acceleration method for some vector sequences.

1. Introduction

There are two well known families of convergence acceleration methods for vector sequences: polynomial methods and epsilon algorithms. Five classical methods have been discussed in [10]: the minimal polynomial extrapolation(MPE) of Cabay and Jackson[2], the reduced rank extrapolation(RRE) of Eddy[3] and Mešina[4], the scalar epsilon algorithm(SEA) and the vector epsilon algorithm(VEA) of Wynn[11,12], and the topological epsilon algorithm(TEA) of Brezinski[1]. In a recent paper [9], Sidi et al. improved the MPE and RRE methods and obtained a modified minimal polynomial extrapolation method(MMPE). Convergence and stability analyses of MPE, RRE, MMPE and TEA are discussed in [7] and [9] respectively. In [14] a rational acceleration method using vector Padé approximation is derived. This method also has the following important properties:

(1) It accelerates the convergence of a slowly converging vector sequence and makes a diverging sequence converge to an "anti–limit" which will be defined in the next section.

(2) It depends only upon the given vector sequence whose convergence is being accelerated; it does not depend on how the vector sequence is generated.

(3) It can use partial components of the vectors to accelerate convergence of the whole vectors.

We call this new method vector Padé approximation method(VPA). In [14] we also gave some properties of VPA, and obtained an algorithm quite similar to the H-algorithm[7]. In this paper, we first introduce vector Padé approximation and an associated acceleration method (i.e., VPA) in **2**. From the viewpoint of Shanks' transformation[6] we will explain the relation between VPA and MMPE. Adopting

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the same technique as in [9], we will analyze the convergence and stability properties of VPA in **3** and **4** respectively. It is easy to see that our conclusions and even the remarks are quite similar to those of [7] and [9]. All the results show that VPA is a bona fide convergence acceleration method.

2. Notations and description of VPA method

Let C^p be the p-dimensional linear complex space, and the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ be defined as usual.

In order to define vector Padé approximation, we introduce the following notations:

$$H_k := \{p(z) : p(z) = \sum_{i=0}^{k} a_i z^i, a_i \in C\},$$

$$E_k := \{e(z) : e(z) = \sum_{i=k+1}^{\infty} a_i z^i, a_i \in C\},$$

$$Z_+^p := \{\vec{l} : \vec{l} = (l_1, \cdots, l_p)^T, \quad l_i \in Z_+, \quad i = 1, \cdots, p\},$$

where p is a given positive integer, and Z_+ is the set of all nonnegative integers. Define $|\vec{l}| = \sum_{i=1}^{p} l_i$, for $\vec{l} \in Z_+^p$,

$$H_n^p := (H_n, \cdots, H_n)^T, \quad E_{\vec{w}}^p := (E_{w_1}, \cdots, E_{w_p})^T.$$

If $g(z) = \sum_{i=0}^{\infty} c_i z^i, c_i \in C$, we denote

$$T_{m,n}^l(g) = \begin{bmatrix} c_l & c_{l-1} & \cdots & c_{l-n+1} \\ c_{l+1} & c_l & \cdots & c_{l-n+2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{l+m-1} & c_{l+m} & \cdots & c_{l+m-n} \end{bmatrix}. \tag{2.1}$$

Here we define $c_i = 0$, if $i < 0$.

Definition. *Let $f(z) = \sum_{i=0}^{\infty} c_i z^i, c_i \in C^p$ be a given power series, $n \in Z_+$ and $\vec{w} = (w_1, \cdots, w_p)^T \in Z_+^p$ be a given integer vector such that*

$$\vec{e} := \vec{w} - n := (w_1 - n, \cdots, w_p - n)^T \in Z_+^p, \qquad |\vec{w}| = p \cdot n + k$$

where k is a given integer. If we can find a vector polynomial $N(z) \in H_n^p$ and a scalar polynomial $M(z) \in H_k$ such that

$$f(z)M(z) - N(z) \in E_{\vec{w}}^p \quad \text{and} \quad M(0) = 1, \tag{2.2}$$

then we call $N(z)M(z)^{-1}$ the $[n, k, \vec{w}]$ vector Padé approximation of f. We denote it as $[n, k, \vec{w}]_f$.

If $H(n, k, \vec{w})$ is nonsingular, then we have (see [13], [14])

$$M(z) = \frac{1}{\det H(n,k,\vec{w})} \begin{vmatrix} 1 & z & \cdots & z^k \\ B(n,k,\vec{w}) & & H(n,k,\vec{w}) & \end{vmatrix}, \tag{2.3}$$

$$N(z) = \frac{1}{\det H(n,k,\vec{w})} \begin{vmatrix} f^{(n)} & z f^{(n-1)} & \cdots & z^k f^{(n-k)} \\ B(n,k,\vec{w}) & & H(n,k,\vec{w}) & \end{vmatrix}, \tag{2.4}$$