Journal of Computational Mathematics, Vol.10, No.1, 1992, 93–97.

ON THE NECESSARY CONDITIONS FOR THE SOLUBILITY OF ALGEBRAIC INVERSE EIGENVALUE PROBLEMS *1)

Xu Shu-fang
(*Computing Center, Academia Sinica, Beijing, China*)

Abstract

In this paper we give some necessary conditions for the solubility of additive inverse eigenvalue problems, multiplicative inverse eigenvalue problems and general inverse eigenvalue problems.

§1. Introduction

In this paper we shall consider the following inverse eigenvalue problems (see [1], [2]).

Problem A (Additive inverse eigenvalue problem). *Given an $n \times n$ Hermitian matrix $A = [a_{ij}]$, and n real numbers $\lambda_1, ..., \lambda_n$, find a real $n \times n$ diagonal matrix $D = \operatorname{diag}(c_1, ..., c_n)$ such that the matrix $A + D$ has eigenvalues $\lambda_1, ..., \lambda_n$.*

Problem M (Multiplicative inverse eigenvalue problem). Given an $n \times n$ positive definite Hermitian matrix $A = [a_{ij}]$, and n positive real numbers $\lambda_1, ..., \lambda_n$, find an $n \times n$ positive definite diagonal matrix $D = (c_1, ..., c_n)$ such that the matrix DA has eigenvalues $\lambda_1, ..., \lambda_n$.

Problem G (General inverse eigenvalue problem). *Given $n+1$ complex $n \times n$ Hermitian matrices $A_0, A_1, ..., A_n$ and n real numbers $\lambda_1, ..., \lambda_n$, find n real numbers $c_1, ..., c_n$, such that the matrix $A(c) = A_0 + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, ..., \lambda_n$.*

A number of sufficient conditions for these problems to have a solution have been discovered (see [1], [3]), but, to our knowledge, only one necessary condition is known and it applies only to Problem A. In the present note we shall give another necessary condition for the solubility of Problem A, which is equivalent to the condition in [4], but the form and the proof of this necessary condition is apparently simple and

* Received October 28, 1988.
1) The Project supported by the National Natural Science Foundation of China.

concise. Then we shall give some necessary conditions for the solubility of Problem M and Problem G.

Notation and Definitions. Throughout this paper we use the following notation. $\mathbb{C}^{m\times l}$ is the set of all $m\times l$ complex matrices. $\mathbb{C}^m$ is the set of all m-dimensional complex column vectors. The norm $\|\ \ \|_F$ stands for Frobenious norm of a matrix. The superscripts T and H are for transpose and conjugate transpose, respectively. I is the $n\times n$ identity matrix, and e_i is the ith column of I. δ_{ij} is the Kronecker delta.

Let k and n be integers, $1\le k\le n$. We use $G_{k,n}$ to denote the set of all increasing sequences of integers,

$$\pi=(j_1,j_2,\cdots,j_k)\ \text{with}\ 1\le j_1<j_2<\cdots<j_k\le n.$$

For arbitrary $\pi=(j_1,...,j_k)\in G_{k,n}$ and $A=[a_{ij}]\in\mathbb{C}^{n\times n}$, we use $A(\pi)$ to denote the $k\times k$ principal submatrix of A whose (i,l) entry is $a_{j_ij_l}(i,l=1,2,\cdots,n)$, $\operatorname{tr}(A)$ to denote the trace of A, and $A^\dagger$ to denote the Moore-Penrose generalized inverse matrix of A. And we define

$$A^{(0)}=A-\operatorname{diag}(a_{11},\cdots,a_{nn}).$$

Without loss of generality we can suppose that $a_{ii}=0, i=1,2,\cdots,n$, in Problem A, $a_{ii}=1, i=1,2,\cdots,n$, in Problem M, and $a_{ii}^{(k)}=\delta_{ik}, k,i=1,2,\cdots,n$, in Problem G, and suppose that $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$ in the three problems.

§2. Main Results

Theorem 1. *The necessary conditions for the solubility of Problem M is*

$$\sum_{1\le i<j\le k}(\lambda_i-\lambda_{j+n-k})^2\ge\lambda_n^2k\max\left\{\|A^{(0)}(\pi)\|_F^2\quad|\pi\in G_{k,n}\right\},\quad 2\le k\le n. \tag{2.1}$$

Theorem 2. *The necessary conditions for the solubility of Problem G is*

$$\sum_{1\le i<j\le k}(\lambda_i-\lambda_{j+n-k})^2\ge k\max\left\{\|A^{(0)}(\pi)+\sum_{i=1}^n x_i(\pi)A_i^{(0)}(\pi)\|_F^2\quad|\pi\in G_{k,n}\right\},$$
$$2\le k\le n, \tag{2.2}$$

where

$$(x_1(\pi),\cdots,x_n(\pi))^T=[S(\pi)]^\dagger b(\pi),$$

$$S(\pi)=\begin{pmatrix} tr(A_1^{(0)}(\pi)A_1^{(0)}(\pi)) & \cdots & tr(A_1^{(0)}(\pi)A_n^{(0)}(\pi))\\ \vdots & \ddots & \vdots\\ tr(A_n^{(0)}(\pi)A_1^{(0)}(\pi)) & \cdots & tr(A_n^{(0)}(\pi)A_n^{(0)}(\pi))\end{pmatrix}, \tag{2.3}$$

$$b(\pi)=(-tr(A_1^{(0)}(\pi)A_0^{(0)}(\pi)),\cdots,-tr(A_n^{(0)}(\pi)A_0^{(0)}(\pi)))^T.$$