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THE ENTROPY CONDITION FOR IMPLICIT TVD SCHEMES*1)

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Abstract

A class of implicit trapezoidal TVD schemes is proven to satisfy a discrete convex entropy inequality and the solution sequence of such implicit trapezoidal schemes converges to the physically relevant solution for genuinely nonlinear scalar conservation laws. The results are extended for a class of generalized implicit one-leg TVD schemes.

§1. Introduction

We consider a hyperbolic conservation law:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x},\quad t>0, \tag{1.1}$$

$$u(x,0)=u^0(x),\quad -\infty<x<\infty.$$

If there exists a convex function $V(u)$ and a differentiable function $F(u)$ such that $V'f'=F'$, the admissible weak solution of (1.1), satisfies

$$\frac{\partial V(u)}{\partial t}+\frac{\partial F(u)}{\partial x}\leq 0 \tag{1.2}$$

in a weak sense, then inequality (1.2) is called the entropy inequality. The pair (V,F) is called the entropy pair. It is well known that the weak solution of (1.1) safisfying the entropy inequality (1.2) for all entropy pairs is unique. The weak admissible solution satisfying the entropy inequality is called the entropy solution.

In the genuinely nonlinear case, where f is, say, strictly convex, if the admissible weak solution of (1.1) satisfies one special convex entropy inequality (1.2), the weak solution is unique (DiPerna [1]).

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The idea of Total Variation Diminishing Schemes was proposed by Harten[2]. The solutions of the conservative TVD scheme have a subsequence which converges to the weak solution of (1.1) in a weak sence. Unfortunately, the limit weak solution is not always unique. Hence, it is necessary to impose an entropy inequality to the TVD scheme to ensure the convergence of the solution sequence to a unique physically relevant solution.

The entropy inequality for first order accurate TVD schemes have been widely studied[3–6,8,9]. Osher[4] showed the entropy inequality for semidiscrete, second order accurate generalized MUSCL schemes. Osher and Chakrayarthy[5] constructed semidiscrete, second order accurate TVD schemes, and proved that those schemes satisfy a semi-discrete entropy inequality. Osher and Tadmor[6] considered the entropy inequality for fully discrete second order accurate explicit TVD schemes. The result given in [6] has the strict CFL limitation. As to first order accurate explicit TVD schemes, the limitation of CFL number is also strict[6,8,9]. Hence, we should consider the implicit TVD scheme to relax the limitation. Our discussion is motivated by papers [4–6, 10].

In Section 2, we give some preliminary results for implicit TVD schemes. In Section 3, we establish the entropy inequality for implicit trapezoidal TVD schemes. In Section 4 and 5, two examples of second order accurate implicit trapezoidal TVD schemes are presented. In the last section we give some explanation about our results.

§2. Preliminary Results on Implicit TVD Schemes

We consider the semidiscrete approximation to (1.1) in conservative form:

$$\frac{du_j}{dt} + \frac{1}{\Delta x}\left(h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}\right) = 0, \tag{2.1}$$

$$u_j(0) = \frac{1}{\Delta x}\int_{\Gamma_j} u^0(x)dx,$$

where

$$h_{j+\frac{1}{2}} = h(u_{j-l+1}, \cdots, u_{j+l}, f, \Delta x), \quad \lambda = \frac{\Delta t}{\Delta x}, \tag{2.2}$$

$$\Gamma_j = \{x : (j - 1/2)\Delta x \le x < (j + 1/2)\Delta x\}.$$

The corresponding discrete entropy inequality is written as

$$\frac{dU(u_j(t))}{dt} + \frac{1}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right) \le 0 \tag{2.3}$$