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AN UNCONDITIONALLY STABLE DIFFERENCE APPROXIMATION FOR A CLASS OF NONLINEAR DISPERSIVE EQUATIONS*

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Abstract

An unconditionally stable leap-frog finite difference scheme for a class of nonlinear dispersive equations is presented and analyzed. The solvability of the difference equation which is a tridiagonal circular linear system is discussed. Moreover, the convergence and stability of the difference scheme are also investigated by a standard argument so that more difficult priori estimations are avoided. Finally, numerical examples are given.

§1. Introduction

We shall consider a leap-frog finite difference approximation of the nonlinear dispersive equation given by

$$u_t + (a(x,t,u))_x + b(x,t,u) - u_{xxt} = 0, \quad (x,t) \in R \times I, \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in R, \tag{1.1b}$$

$$u(x+1,t) = u(x,t), \quad (x,t) \in R \times I, \tag{1.1c}$$

where $I = [0,T](T > 0)$, and R is the real line. The coefficients a and b in (1.1a) will be C^1 functions defined on $R \times I \times R$, 1-periodic with respect to their first argument. u_0 in (1.1b) is also g given 1-periodic function.

Theoretical results about the existence, uniqueness and regularity for (1.1) can be found in [1,2] and the references contained therein. Numerical approximations of (1.1) based on the finite element method[1], the finite difference method[3] and the spectral method[4] have also been considered. W.H. Ford and T.W. Ting[3] have studied the convergence and stability of the Crank-Nicolson scheme, but the Crank-Nicolson scheme is a nonlinear system and is hard to solve. Some physicists and engineers proposed some finite element and finite difference schemes, but they did not get the proof of convergence and stability (see [5, 6]).

In this paper we devote a leap-frog finite difference scheme which is a tridiagonal circular linear system and can be easily solved by the Seidel iteration method. Using the standard argument[1,7], we prove its convergence and stability. Therefore we can avoid quite difficult priori estimations.

Throughout this paper, we assume that (1.1) has a unique smooth solution U defined in $R \times I$. The letter C will be used to indicate generic constants, and the usual functional notation will be employed to specify dependence.

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§2. Some Symbols and Finite Difference Scheme

We introduce a grid $x_j = jh = (0 \le j \le J), h = 1/J, t^n = nk (0 \le n \le N)$, where J and N are positive integers, h and k are space-step length and time-step length respectively.

In the following we often use superscript n to denote the n-th time level. We set the following difference operators:

$$(u_j^n)_x = (u_{j+1}^n - u_j^n)/h, \ (u_j^n)_{\bar{x}} = (u_j^n - u_{j-1}^n)/h, \ (u_j^n)_{\hat{x}} = (u_{j+1}^n - u_{j-1}^n)/(2h),$$

where u_j^n is the approximation of $U(jh, nk)$. Similarly, we can define difference operators $(u_j^n)_t$, $(u_j^n)_{\bar{t}}$ and $(u_j^n)_{\hat{t}}$.

If V and W are 1-periodic grid functions, we denote by v_j and w_j their values at x_j respectively. Set

$$(V, W) = h \sum_{j=1}^{J} v_j \bar{w}_j, \quad (V, W)_1 = (V, W) + (V_x, W_x),$$
$$\|V\|^2 = (V, V), \qquad \|V\|_1^2 = (V, V)_1,$$

where $V_x = (v_{jx})$ and $W = (w_{jx})$ are difference grid functions.

With these notations we can consider the following leap-frog difference scheme of (1.1):

$$u_{j\hat{t}}^n + (a_j^n(u_j^n))_{\hat{x}} + b_j^n(u_j^n) - u_{jx\bar{x}\hat{t}}^n = 0; \quad 1 \le j \le J, 1 \le n \le N, \tag{2.1a}$$
$$u_{jt}^0 + (a_j^0(u_j^0))_{\hat{x}} + b_j^0(u_j^0) - u_{jx\bar{x}t}^0 = 0, \quad 1 \le j \le J, \tag{2.1b}$$
$$u_j^0 = u_0(jh), \quad 1 \le j \le J, \tag{2.1c}$$
$$u_{j+rJ}^n = u_j^n, \quad 1 \le j \le J, \ 0 \le n \le N, \ r = \pm 1, \pm 2, \cdots, \tag{2.1d}$$

where $a_j^n(u_j^n) = a(jh, nk, u_j^n)$ and $b_j^n(u_j^n) = b(jh, nk, u_j^n)$.

§3. Solvability, Convergence and Stability of the Difference Solution

First of all, we discuss the solvability of (2.1). Note that (2.1a) and (2.1b) can be rewritten as

$$-u_{j-1}^{n+1} + (2 + h^2) u_j^{n+1} - u_{j+1}^{n+1} = b_j^n, \quad 1 \le j \le J, \ 0 \le n \le N - 1, \tag{3.1}$$

where $b_j^n = h^2 u_j^{n-1} - h^2 (u_j^{n-1})_{x\bar{x}} - 2kh^2[(a_j^n(u_j^n))_{\hat{x}} + b_j^n(u_j^n)]$ $(1 \le n \le N-1)$ and $b_j^0 = h^2 u_j^0 - h^2 (u_j^0)_{x\bar{x}} - kh^2[(a_j^0(u_j^0))_{\hat{x}} + b_j^0(u_j^0)]$.

The coefficient matrix of the tridiagonal circular linear system (3.1) is nonsingular because it is strictly row-wise diagonally dominant. So the following result holds:

Theorem 3.1. *Difference equation* (2.1) *is always solvable.*

On the convergence and stability of the solution of (2.1), we have