Journal of Computational Mathematics, Vol.9, No.1, 1991, 86-96.

SYMPLECTIC DIFFERENCE SCHEMES FOR HAMILTONIAN SYSTEMS IN GENERAL SYMPLECTIC STRUCTURE*1)

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Abstract

We consider the construction of phase flow generating functions and symplectic difference schemes for Hamiltonian systems in general symplectic structure with variable coefficients.

§1. Introduction

The standard symplectic structure w on $\mathbb{R}^{2n}$ is of the form

$$w=\sum_{i<j} J_{ij}dz_i\wedge dz_j,\quad J=\begin{bmatrix} 0 & I_n \\ -I_n & 0\end{bmatrix}. \tag{1}$$

On such symplectic manifold, the Hamiltonian system has the simplest form

$$\frac{dz}{dt}=J^{-1}\Delta H(z) \tag{2}$$

with the Hamiltonian $H(z)$. The phase flow of the Hamiltonian system (2) preserves the symplectic structure (1). Therefore it preserves phase areas and the phase volume of the phase space. Feng Kang et al. in [2-4] developed a generating function method to construct systematically symplectic difference schemes with arbitrary order of accuracy to approximate the system (2). The transition of such difference schemes from one time-step to the next is a symplectic mapping. So they preserve the symplectic structure. It leads to the preservations of phase areas and the phase volume of the phase space.

Generally, a general symplectic structure on $\mathbb{R}^{2n}$ with variable coefficients

$$w=\sum_{i<j} K_{ij}(z)dz_i\wedge dz_j, \tag{3}$$

which is a non-degenerate, colsed 2-form. In this case, the Hamiltonian system of the form

$$\frac{dz}{dt}=K^{-1}(z)\nabla H(z),\quad K_{ji}=-K_{ij}, \tag{4}$$

* Received March 1, 1988.
1) The Project Supported by National Natural Science Foundation of China.

where $H(z)$ is the Hamiltonian. As above, the phase flow of the system (4) preserves the symplectic structure (3). In numerical simulation for (4), usual discretization can not preserve the symplectic structure (3). By Darboux's theorem, we can, of course, transform (4) into (2) and then use the method in [2–4]. But (i) it is difficult to find out the transformation, and (ii) it is also interesting to discretize (4) directly such that the transition preserves the symplectic structure (3).

In this paper, we try to construct symplectic difference schemes in same way as in [4]. In [4], the key point is to introduce a linear transformation from the symplectic manifold $(\boldsymbol{R}^{4n}, \tilde{J}_{4n})$ into the symplectic manifold $(\boldsymbol{R}^{4n}, J_{4n})$ which transforms the $\tilde{J}_{4n}$-Lagrangian submanifold into J_{4n}-Lagrangian submanifold. Of course the inverse transformation transforms the J_{4n}-Lagrangian submanifolds into the $\tilde{J}_{4n}$-Lagrangian submanifolds. In fact, we can also take nonlinear transformations for the same purpose. This was first noted by Feng Kang and has been used in [5]. For other related developments, see [6–15]. In this paper, we use nonlinear transformations to reach our purpose.

In Section 2, we give the relationship between the $K(z)$-symplectic mappings and the gradient mappings. In Section 3, we consider the generating functions of the phase flow of the Hamiltonian system (4) and the corresponding Hamilton-Jacobi equation. When the Hamiltonian function is analytic, then the generating function can be expanded as a power series in t and its coefficients can be recursively determined. With the aid of such an expression, in Section 4 we give a systematic method to construct $K(z)$-symplectic difference schemes with arbitrary order of accuracy.

We shall only consider the local case throughout the paper.

§2. Generating Functions for $K(z)$-Symplectic Mappings

Let $\boldsymbol{R}^{2n}$ be a $2n$-dimensional real space. The elements of $\boldsymbol{R}^{2n}$ are $2n$-dimensional column vectors $z = (z_1, \cdots, z_n, z_{n+1}, \cdots, z_{2n})^T$. The superscript T represents the matrix transpose.

A symplectic form w on $\boldsymbol{R}^{2n}$ is a non-degenerate, closed 2-form, defined by

$$w = \frac{1}{2}\sum_{i,j=1}^{2n} K_{ij}(z)dz_i \wedge dz_j. \tag{5}$$

Denote the entries of $K(z)$ by $K_{ij}(z)$, $i,j = 1,\cdots,2n$. Then $K(z)$ is anti-symmetric. The non-degeneracy and closedness of w imply that $\det K(z) \neq 0$ and $K_{ij}(z)$ is subject to the condition

$$\frac{\partial K_{ij}(z)}{\partial z_l} + \frac{\partial K_{jl}(z)}{\partial z_i} + \frac{\partial K_{li}(z)}{\partial z_j} = 0, \qquad i,j,l = 1,\cdots,2n. \tag{6}$$

From now on, we always identify the symplectic form w with $K(z)$.

Denote

$$J_{4n} = \begin{bmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{bmatrix}, \quad \tilde{K}(\hat{z}, z) = \begin{bmatrix} K(\hat{z}) & 0 \\ 0 & -K(z) \end{bmatrix}.$$

Evidently they define two symplectic structures on $\boldsymbol{R}^{4n}$:

$$\Omega = \sum_{i=1}^{2n} dw_i \wedge dw_{2n+i}, \quad \tilde{\Omega} = \frac{1}{2}\sum_{i,j=1}^{2n}(k_{ij}(\hat{z})d\hat{z}_i \wedge d\hat{z}_j - k_{ij}(z)dz_i \wedge dz_j).$$