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A DIRECT METHOD FOR PITCHFORK BIFURCATION POINTS*

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Abstract

To overcome the difficulty caused by the singularity at the pitchfork bifurcation points, we introduce the homotopy parameter so that the problem of computing the pitchfork bifurcation points can be transferred to that of computing the fold points of degree 3 with respect to the homotopy parameter. An extended system for pitchfork bifurcation points is given. The regularity of the extended system is proved. Finally, the numerical examples show the effectiveness of our method.

§1. Introduction

The pitchfork bifurcation is one of the basic bifurcation phenomena in nonlinear problems. We consider the following nonlinear problem with one parameter

$$f(\lambda, x) = 0, \tag{1.1}$$

where λ is a real parameter, $x \in X$, a Banach space, f is a C^3 Fredholm operator with index 0 from $R \times X$ to X. To determine the bifurcation points from the trivial solution we only need computing the eigenvalues of the corresponding linearized operator in the nonlinear problems. It brings us difficulties to compute the pitchfork bifurcation points from the non-trivial solutions, which are unknown in advance. What Concerns us in this paper is to compute such kind of pitchfork bifurcation points, which are called the secondary pitchfork bifurcation points, in the nonlinear problem (1.1).

Several extended systems for computing the secondary bifurcation points have been proposed in [1]–[3]. We introduce a homotopy parameter so that the problem of computing the pitchfork bifurcation points can be transferred to the problem of computing the fold points of degree 3 with respect to the homotopy parameter, which can be solved by using our algorithm in [4]. In Section 3 an extended system for pitchfork bifurcation points is given and its regularity is proved. Finally, in Section 4 we give numerical examples to show the effectiveness of our method.

§2. Pitchfork Bifurcation Point

Let $f_\lambda^0, f_x^0, f_{\lambda\lambda}^0, f_{\lambda x}^0, f_{xx}^0, f_{xxx}^0, \cdots$ denote the partial Fréchet-derivatives of f at $a_0 = (\lambda_0, x_0)$. Denote the dual pairing of $x \in X$ and $\psi \in X^*$ by $\langle \psi, x \rangle$, where X^* is the conjugate space of X.

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Definition 2.1. *A point $a_0 = (\lambda_0, x_0)$ is bifurcation point of (1.1) with respect to λ if*

$$f(a_0) = 0, \tag{2.1}$$
$$\text{Dim Ker } f_x(a_0) \geq 1, \tag{2.2}$$
$$f_\lambda^0 \in \text{ Range } f_x^0, \tag{2.3}$$

where Ker f_x^0 *is the unll space of Jacobian* f_x^0.

We always assume that

$$\text{Dim Ker } f_x^0 = \text{Codim Range} f_x^0 = 1. \tag{2.4}$$

In this case there exist nontrivial $\phi_0 \in X$ and $\psi_0 \in X^*$ such that

$$N = \text{Ker} f_x^0 = \{r\phi_0 | r \in R\}, \tag{2.5}$$
$$M = \text{Range} f_x^0 = \{x \in X| < \psi_0, x >= 0\}. \tag{2.6}$$

We can decompose $X = N \otimes V_0$ where V_0 is a complement of N in X. (2.3) in Definition 2.1 implies $< \psi_0, f_\lambda^0 >= 0$ and there is a unique $v_0 \in V_0$ such that

$$f_x^0 v_0 + f_\lambda^0 = 0. \tag{2.7}$$

Definition 2.2. *A bifurcation point a_0 of (1.1) with respect to λ is a pitchfork bifurcation point of (1.1) with respect to λ if*

$$< \psi_0, f_{xx}^0 \phi_0 \phi_0 >= 0, \tag{2.8}$$
$$< \psi_0, f_{\lambda x}^0 \phi_0 + f_{xx}^0 \phi_0 v_0 > \neq 0, \tag{2.9}$$
$$< \psi_0, 3 f_{xx}^0 \phi_0 u_0 + f_{xxx}^0 \phi_0 \phi_0 \phi_0 > \neq 0, \tag{2.10}$$

where ϕ_0, v_0 are given in (2.5), (2.7), and $u_0 \in V_0$ is uniquely determined by

$$f_x^0 u_0 + f_{xx}^0 \phi_0 \phi_0 = 0. \tag{2.11}$$

§3. Regular extended system

We introduce a homotopy path of (1.1):

$$F(\lambda, \mu, x) = f(\lambda, x) - \mu f(\lambda^*, x^*) = 0, \tag{3.1}$$

where μ is a homotopy parameter. In probability 1 we can choose such λ^*, x^*, that $f(\lambda^*, x^*)$ $\bar{\in}$ Range f_x^0 i.e. $< \psi_0, f(\lambda^*, x^*) > \neq 0$ (see [5]).

We propose an extended system for the pitchfork bifurcation points of (1.1) with respect to λ as follows:

$$\begin{cases} f(\lambda, x) - \mu f(\lambda^*, x^*) = 0, \quad f_x \phi = 0, \\ l\phi - 1 = 0, \quad f_{xx}\phi\phi + f_x u = 0, \quad lu = 0, \end{cases} \tag{3.2}$$

where $l \in X^*$. We take such l that $lx = 0$ means $x \in V_0$ and $l\phi_0 = 1$. Usually, lw can be taken as the r-th component of w for the actual computation. Next, we are going to