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POLYNOMIAL OF DEGREE FOUR INTERPOLATION ON TRIANGLES*

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Abstract

A method for constructing the C^1 piecewise polynomial surface of degree four on triangles is presented in this paper. On every triangle, only nine interpolation conditions, which are function values and first partial derivatives at the vertices of the triangle, are needed for constructing the surface.

§1. Introduction

In finite-element analysis, surface design and other fields, constructing an interpolation surface on triangles is often needed. In many cases, it is necessary that the surface constructed be C^1 continuous, and it is desirable that the surface constructed is a polynomial and the degree of the polynomial is as low as possible because a polynomial is simple in construction and easy to calculate. Now there are mainly three methods for constructing C^1 continuous piecewise polynomial surfaces on triangles. On every triangle, the first method needs 21 inrerpolation conditions, and the surface constructed is a polynomial of degree five. The difficulty in using the first method lies in how to find the second partial derivatives. The high degree of the surface constructed is also unfavorable. Other two methods in [1] and [2] need 12 interpolation conditions on every triangle; the surfaces constructed on every triangle are C^1 continuous piecewise cubic polynomial and quadric one respectively. The shortage of the methods in [1] and [2] is that every triangle must be subdivided. The method in [3] is suitable for constructing a C^1 polynomial surface interpolating scattered data points, but the degree of the surface constructed is seven.

The method presented in this paper needs only nine interpolation conditions on every triangle, which are function values and first partial derivatives at the vertices of the triangle. The surface constructed on every triangle is a polynomial piece of degree four. The whole C^1 surface is all composed of polynomial pieces of degree four. The limitation of the method in this paper is that the degrees of all the interior vertices of the triangulation are odd.

§2. Basic Ideas

If a vertex P_i is shared by m triangles, then the degree of P_i is m (the degree of P_i in Fig.1 is 5).

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Suppose that the degree of all vertices (except those on the boundary of the interpolation region) are odd. At every vertex $P_i (i = 1, 2, \cdots)$, the function values and first partial derivatives $\{F_i, (F_x)_i, (F_y)_i\}$ are given. The procedure of constructing an interpolation surface in this paper is as follows:

(1) On every triangle, a cubic polynomial surface piece which satisfies nine given interpolation conditions is constructed. The surface has an undefined parameter.

(2) Around every vertex of the triangulation, a C^1 continuous piecewise cubic polynomial surface piece is constructed by the surface pieces, each of which has an undefined parameter.

(3) On every triangle, a polynomial surface piece of degree four is formed by weighting the three C^1 continuous piecewise cubic polynomial surface pieces around the vertices of the triangle. The whole surface is all composed of polynomial surface pieces of degree four on the triangles.

§3. Concret Realization of the Method

(1) Construction of a surface piece having an undefined parameter on a triangle.

Suppose that T is a triangle with vertices $P_\beta = \{(x_\beta, y_\beta)\}, \beta = i, j, k$ and its number is t; see Fig.3. At the three vertices of T, there are altogether nine interpolation conditions $\{F_\beta, (F_x)_\beta, (F_y)_\beta\}$, $\beta = i, j, k$. Let $(N_t)_\beta$ be the unit normal vector of the opposite side of vertex P_β, and $((L_t)_i, (L_t)_j, (L_t)_k)$ be the area coordinates of T. Then

$$\left.\begin{aligned}
(L_t)_i &= (L_t(x,y))_i = [x_j y_k - x_k y_j + (y_j - y_k)x + (x_k - x_j)y]/(2S_t),\\
(L_t)_j &= (L_t(x,y))_j = [x_k y_i - x_i y_k + (y_k - y_i)x + (x_i - x_k)y]/(2S_t),\\
(L_t)_k &= (L_t(x,y))_k = [x_i y_j - x_j y_i + (y_i - y_j)x + (x_j - x_i)y]/(2S_t),
\end{aligned}\right\} \tag{3.1}$$

where S_t is the area of T.

The cubic polynomial surface piece, which has an undefined parameter, on T is

$$\begin{aligned}
F_t(x,y) = {} & (F_i + A_{ij}(L_t)_j + A_{ik}(L_t)_k)(L_t)_i^2 + (F_j + A_{jk}(L_t)_k + A_{ji}(L_t)_i)(L_t)_j^2\\
& + (F_k + A_{ki}(L_t)_i + A_{kj}(L_t)_j)(L_t)_k^2 + (A_t)_p (L_t)_i (L_t)_j (L_t)_k,
\end{aligned} \tag{3.2}$$

where

$$\left.\begin{aligned}
A_{ij} &= 2F_i + (F_x)_i(x_j - x_i) + (F_y)_i(y_j - y_i),\\
A_{ik} &= 2F_i + (F_x)_i(x_k - x_i) + (F_y)_i(y_k - y_i),\\
A_{jk} &= 2F_j + (F_x)_j(x_k - x_j) + (F_y)_j(y_k - y_j),\\
A_{ji} &= 2F_j + (F_x)_j(x_i - x_j) + (F_y)_j(y_i - y_j),\\
A_{ki} &= 2F_k + (F_x)_k(x_i - x_k) + (F_y)_k(y_i - y_k),\\
A_{kj} &= 2F_k + (F_x)_k(x_j - x_k) + (F_y)_k(y_j - y_k).
\end{aligned}\right\} \tag{3.3}$$

It can be verified that $f_t(x, y)$ takes the nine given interpolation conditions $\{F_\beta, (F_x)_\beta, (F_y)_\beta\}$, $\beta = i, j, k$. $(A_t)_p$ is an undefined parameter.

(2) Construction of a surface piece $F_i(x, y)$ around vertex P_i.

For vertex P_i without losing generality, suppose that its degree is 5; see Fig. 2. The surface pieces corresponding to (3.2) on the five triangles are $F_1(x, y), F_2(x, y), \cdots, F_5(x, y)$