Journal of Computational Mathematics, Vol.9, No.3, 1991, 238–246.

NON-CLASSICAL ELLIPTIC PROJECTIONS AND L^2-ERROR ESTIMATES FOR GALERKIN METHODS FOR PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract

In this paper we shall define a so-called "non-classical" elliptic projection associated with an integro-differential operator. The properties of this projection will be analyzed and used to obtain the optimal L^2 error estimates for the continuous and discrete time Galerkin procedures when applied to linear integro-differential equations of parabolic type.

§1. Introduction

Let Ω be an open bounded subset in $R^n (n \geq 1)$ with smooth boundary $\partial\Omega$ and consider the following integro-differential equation of parabolic type:

$$\rho(x)u_t(x,t) = \nabla \cdot [a(x)\nabla u(x,t)] + \int_0^t \nabla \cdot [b(x,t,\tau)\nabla u(x,\tau)]d\tau + f(x,t), \quad \text{in } Q_T, \qquad (1.1)$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \qquad (1.2)$$

$$u(x,t) = 0, \quad \text{on} \quad S_T = \partial\Omega \times [0,T], \qquad (1.3)$$

where $Q_T = \Omega \times (0,T], T > 0; \nabla$ is the gradient operator in $R^n; \rho(x), a(x), b(x,t,\tau)$ and $f(x,t)$ are known functions which are assumed to be as smooth as needed throughout this paper. In addition, we assume that there exist two positive constants c_*, c^* such that

$$0 < c_* \leq \rho(x), \quad a(x) \leq c^*, \quad x \in \Omega. \qquad (1.4)$$

Recently, some attention has been given to numerical approximations to the solution of (1.1)–(1.3). Sloan and Thomée[15] considered the time discretization approximations, Cannon and Li and Lin[4] have formulated a Galerkin procedure for general linear equations. Optimal L^2 error estimates in the case when $a(x) = 1$ and $b(x,t,\tau) = b(t,\tau)$ appear in [11]. The problems of existence, uniqueness and stability of the solution can be found in [9, 12, 13, 17].

When $a(x) \neq$ const. and $b(x,t,\tau)$ is dependent upon x, the method developed in [11] fails to provide the desired L^2 error estimates. The main reason of this is that there are two second order operators on the right-hand side of (1.1), so the usual elliptic projection method discovered by Wheeler in [18] does not work in this case in general. This suggests that we need to treat the operator

$$\nabla \cdot [a(x)\nabla u] + \int_0^t \nabla \cdot [b(x,t,\tau)\nabla u(x,\tau)]d\tau \qquad (1.5)$$

* Received May 6, 1988.

as a single unit. In this paper we shall define a "non-classical" elliptic projection suitable for (1.5). In the special case when $b = 0$, our new projection reduces to the usual elliptic projection defined in [18].

Let $H^s(\Omega)$ denote Sobolev spaces on Ω and $\|\cdot\|_s$ the related norm, with $H^0(\Omega) = L^2(\Omega)$ with norm $\|\cdot\|$. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_1$.

Let $\{S_h\}_{0<h\leq 1}$ be the finite-dimensional subspaces in H_0^1 which satisfy the following approximation property:

$$\inf_{\chi\in S_h}\{\|v-\chi\| + h\|v-\chi\|_1\} \leq Ch^s\|v\|_s, \quad v\in H^s\cap H_0^1, \quad s\geq 1, \tag{1.6}$$

where C is a positive constant independent of h and $v \in H^s \cap H_0^1$.

If X is a normed space with norm $\|\cdot\|_X$ and $\phi : [0,T] \to X$, we define

$$\|\phi\|^2_{L^2(X)} = \int_0^T \|\phi(t)\|^2_X dt, \quad \|\phi\|_{L^\infty(X)} = \text{ess} \sup_{0\leq t\leq T} \|\phi(t)\|_X.$$

The continuous Galerkin approximation to the solution u of (1.1)–(1.3) is defined to be a map $U(t) : [0,T] \to S_h$ such that

$$(\rho U_t, \chi) + \Big(a\nabla U + \int_0^t b\nabla U(\tau)d\tau, \nabla\chi\Big) = (f,\chi), \quad t>0, \quad \chi\in S_h, \tag{1.7}$$

$$U(0,) - u_0 \text{ small}, \tag{1.8}$$

where

$$(\phi,\psi) = \int_\Omega \phi(x)\psi(x)dx$$

for scalar and vector functions, respectively. The choice of $U(0)$ will be described later. We know that (1.7)–(1.8) is actually a system of ordinary integro-differential equations and it can be easily checked that for any $U(0) \in S_h$ there exists a unique $U(t)$ for $t > 0$.

Let N be a positive integer, $\Delta t = T/N, t_m = m\Delta t$ and $t_{m+1/2} = (m+1/2)\Delta t$; then we define $f_m = f(t_m)$and $f_{m+1/2} = (1/2)(f_{m+1} + f_m)$. For $t(\tau), g(\tau)$ smooth, we know that

$$\int_{t_k}^{t_{k+1}} f(\tau)g(\tau)d\tau = \Delta t f(t_{t+1/2})g_{k+1/2} + \varepsilon_k(f,g).$$

Since it is easy to verify

$$\int_{t_k}^{t_{k+1}} fg d\tau = \Delta t(fg)_{k+1/2} + \frac{1}{2}\int_{t_k}^{t_{k+1}} (t_{k+1}-\tau)(t_k-\tau)\frac{d^2(fg)}{d\tau^2}d\tau,$$

$$(fg)_{k+1/2} = f_{k+1/2}g_{k+1/2} + \frac{1}{4}\Big(\int_{t_k}^{t_{k+1}} \frac{df}{d\tau}d\tau\Big)\Big(\int_{t_k}^{t_{k+1}} \frac{dg}{d\tau}\Big)d\tau,$$

$$f_{k+1/2} = f(t_{k+1/2}) + \frac{1}{2}\Big[\int_{t_{k+1/2}}^{t_{k+1}} (t_{k+1}-\tau)\frac{d^2f}{d\tau^2}d\tau + \int_{t_{k+1/2}}^{t_k} (t_k-\tau)\frac{d^2f}{d\tau^2}d\tau\Big],$$

we see that the error $\varepsilon_k(f,g)$ can be represented by

$$\varepsilon_k(f,g) = \frac{1}{2}\int_{t_k}^{t_{k+1}} (t_{k+1}-\tau)(t_k-\tau)\frac{d^2(fg)}{d\tau^2}d\tau + \frac{\Delta t}{4}\Big(\int_{t_k}^{t_{k+1}} \frac{df}{d\tau}d\tau\Big)\Big(\int_{t_k}^{t_{k+1}} \frac{dg}{d\tau}\Big)d\tau$$
$$+\frac{\Delta t}{2}\Big[\int_{t_{k+1/2}}^{t_{k+1}} (t_{k+1/2}-\tau)\frac{d^2f}{d\tau^2}d\tau + \int_{t_{k+1/2}}^{t_k} (t_k-\tau)\frac{d^2f}{d\tau^2}d\tau\Big]g_{k+1/2}.$$