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A CLASS OF MULTISTEP METHOD CONTAINING SECOND ORDER DERIVATIVES FOR SOLVING STIFF ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract

In this paper a general k-step k-order multistep method containing derivatives of second order is given. In particular, a class of k-step $(k+1)th$-order stiff stable multistep methods for $k = 3 - 9$ is constructed. Under the same accuracy, these methods are possessed of a larger absolute stability region than those of Gear's [1] and Enright's [2]. Hence they are suitable for solving stiff initial value problems in ordinary differential equations.

§1. Introduction

For the initial value problem of first order ordinary differential equations

$$y' = f(x,y), \quad y(a) = \eta, \quad a \le x \le b, \tag{1}$$

we consider a k-step method which contains derivatives of second order

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}, \tag{2}$$

where $y_{n+i}, i = 0, 1, \cdots, k$, are numerical approximate solutions of solution $y(x)$ of (1) at $x = x_{n+i}$; further $y'_{n+i} = f(x_{n+i}, y_{n+i}), y''_{n+i} = f'_{n+i} = f'(x_{n+i}, y_{n+i})$. The characteristic polynomials of (2) are

$$\rho(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i, \quad \sigma(\xi) = \sum_{i=0}^{k} \beta_i \xi^i, \quad \gamma(\xi) = \sum_{i=0}^{k} \gamma_i \xi^i.$$

Our purposes are: (i) construct a k-step, k-order method and give its error constants, (ii) construct for $k = 3 - 9$ a class of k-step, $(k+1)th$-order stiff stable method, which can be automatically generated by computer. For the same accuracy, the stability region of the methods in this paper is obviously larger than those of Gear's [1] and Enright's [2]. So they are suitable for solving initial value problems of stiff ordinary differential equations.

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§2. Construction of the Methods

Definition. *The method (2) is of order p, if for arbitrary function $y(x) \in C^{p+1}_{[a,b]}$, the following relation holds;*

$$L[y(x);h] = \sum_{i=0}^{k} \alpha_i y(x+ih) - h\sum_{i=0}^{k} \beta_i y'(x+ih) - h^2 \sum_{i=0}^{k} \gamma_i y''(x+ih)$$
$$= C_{p+1} h^{p+1} y^{(p+1)}(x) + O(h^{p+2}), \quad h \to 0, \tag{3}$$

where p and C_{p+1} are independent of $y(x)$, and $C_{p+1} \neq 0$ [3].

Lemma. *Let E be a shift operator such that $Ef(x) = f(x+h)$, and $\nabla f(x) = f(x) - f(x-h)$, $\Delta f(x) = f(x+h) - f(x)$, $Df(x) = \dfrac{df(x)}{dx}$. Suppose the function $f(x)$ is sufficiently differentiable. Then, the relations*

$$hD = \sum_{j=1}^{\infty} \frac{1}{j}\nabla^j, \quad (hD)^2 = \sum_{j=2}^{\infty}\sum_{i=1}^{j-1} \frac{1}{i(j-i)}\nabla^j \tag{4}$$

hold.

Proof. Because

$$E = e^{hD} = (1+\Delta) = (1-\nabla)^{-1}, \tag{5}$$

it follows, that

$$hD = -\ln(1-\nabla) = \sum_{j=1}^{\infty} \frac{1}{j}\nabla^j, \quad (hD)^2 = \sum_{j=2}^{\infty}\sum_{i=1}^{j-1} \frac{1}{i(j-i)}\nabla^j.$$

Theorem. *Suppose the coefficients in the k-step method*

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i} \tag{6}$$

are

$$\alpha_{k-m} = (-1)^m \sum_{i=0}^{k} \left[\beta_i A_1^{(k-i)} + \sum_{j=2}^{k} (\beta_i A_j^{(k-i)} + \gamma_i B_j^{(k-i)}) \binom{j}{m}\right], \quad m = 0, 1;$$
$$\alpha_{k-m} = (-1)^m \sum_{j=m}^{k}\sum_{i=0}^{k} (\beta_i A_j^{(k-i)} + \gamma_i B_j^{(k-i)}) \binom{j}{m}, \quad m = 2(1)k. \tag{7}$$

Then the method (6) is of order k and possesses an error constant

$$C = \frac{1}{\sigma(1)} \sum_{i=0}^{k} (\beta_i A_{k+1}^{(k-i)} + r_i B_{k+1}^{(k-i)}), \tag{8}$$

where

$$A_j^{(l)} = \nabla^l A_j^{(0)}, \quad A_j^{(0)} = \frac{1}{j}, \quad j = 1(1)k;$$
$$B_j^{(l)} = \nabla^l B_j^{(0)}, \quad B_j^{(0)} = \sum_{i=1}^{j-1} \frac{1}{i(j-i)}, \quad j = 2(1)k, \ l = 0(1)k; \tag{9}$$
$$A_\nu^{(0)} = B_\nu^{(0)}, \quad \nu < 0.$$