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A DECOMPOSITION METHOD FOR SOME BIHARMONIC PROBLEMS*

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Abstract

In this paper we consider two biharmonic problems [13] which will be conventionally indicated as "simply supported" and "clamped plate" problem.

We construct a decomposition method [16], [19] related to the partition of the plate in two, or more, subdomains. We carry on the numerical treatment of the method first decoupling these fourth order problems into two second order problems, then discretizing these problems by mixed linear finite element and obtaining an algebraic system. Moreover we present an iterative block algorithm for solving the foregoing system, which can be efficiently developed on parallel computers.

At the end we extend the method to the respective biharmonic variational inequalities [10].

Notations

Let:

— Ω be a bounded open set of R^2, whose boundary $\Gamma = \partial\Omega$ is "sufficiently" regular [8];

— $\Omega = \Omega_1 \cup \gamma \cup \Omega_2$, where $\Omega_i (i = 1, 2)$ are sufficiently regular open sets staying in the opposite sides of a regular curve γ;

— $\omega \subset \Omega$ be any regular open set containing γ. We will call ω "lacing" set of Ω_1 and Ω_2 owing to raisons that will be clear in future developments;

— $\Gamma_i = \partial\Omega_i$;

— γ_m be the trace operator $\partial_n^m\cdot$ defined on the boundary of an open set, n being the external normal vector [8].

We consider the following functional spaces:

(1) $H := L^2(\Omega)$ endowed with the scalar product $(u,v)_{0,\Omega} = \int_\Omega uv dx$ and the associated norm;

(2) $H_1 := \{v : v \in H / v = 0$ a.e. in $\Omega_{i+1}\} (i = 1, 2)$ where the indexes are counted modulo 2 when necessary;

(3) $V := H^1(\Omega)$ with the scalar product $(u,v)_V = \int_\Omega (\text{grad } u \cdot \text{grad } v + uv)$;

(4) $V_i := \{v : v \in V / v = 0$ a.e. in $\Omega_{i+1}\}, i = 1, 2$;

(5) $\overset{\circ}{V} := H_0^1(\Omega)$ with the scalar product $(u,v)_{1,\Omega} = \int_\Omega \text{grad } u \cdot \text{grad } v dx$;

(6) $\overset{\circ}{V}_i := \{v : v \in V / v = 0$ a.e. in $\Omega_{i+1}\}, i = 1, 2$;

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(7) $\overset{\circ}{V}_3 := \{v : v \in \overset{\circ}{V} / v = 0 \text{ a.e. in } \Omega_i \backslash \omega,\ i = 1,2\}$;

(8) $W := H^2(\Omega) \cap \overset{\circ}{V}$ or $W = H_0^2(\Omega)$ endowed with the scalar product $(u,v)_W = \int_\Omega \Delta u \cdot \Delta v dx$, where $\Delta\cdot = \partial_{11} \cdot + \partial_{22} \cdot$ is the harmonic operator;

(9) $W_i := \{v : v \in W / v = 0 \text{ in } \Omega_{i+1}\}, i = 1,2$;

(10) $W_3 := \{v : v \in W / v = 0 \text{ in } \Omega_i \backslash \omega (i = 1,2)\}$.

Remark I. We remark the following obvious relations [20]:

(11) $V = V_1 + \overset{\circ}{V}_3 + V_2$;

(12) $\overset{\circ}{V} = \overset{\circ}{V}_1 + \overset{\circ}{V}_3 + \overset{\circ}{V}_2$;

(13) $W = W_1 + W_3 + W_2$.

First Part

1. A Minimum Problem in $W = H^2(\Omega) \cap \overset{\circ}{V}$. The Euler Equation. The Simply Supported Plate Problem.

We consider the following potential energy functional for a simply supported plate [13]:

$$F(v) = \frac{1}{2}|v|_W^2 - (f,v)_{0,\Omega}, \quad v \in W,\ f \in H. \tag{1.1.1}$$

It is well known that F is weakly lower semicontinuous, Frechet differentiable, strictly convex, coercive and that it results [8]:

$$< \text{grad } F(u), v >_W = (u,v)_W - (f,v)_{0,\Omega}, \quad \forall v \in W \tag{1.1.2}$$

where the symbol $< \circ, \circ >_W$ denotes the duality pairing between W and the dual space W'.

As a consequence of the above quoted properties we conclude that the extreme problem:

$$\inf_{v \in W} F(v) \tag{1.1.3}$$

has a unique minimum point u:

$$F(u) = \min_{v \in W} F(v) \tag{1.1.4}$$

which corresponds to the unique solution of the following W.elliptic variational problem:

$$u \in W : (u,v)_W = (f,v)_{0,\Omega}, \quad \forall v \in W. \tag{1.1.5}$$

As a consequence of (1.1.5) the strong formulation of the simply supported plate problem is, in formal way, the following:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ \gamma_0 u|_\Gamma = \gamma_0 \Delta u|_\Gamma = 0 \end{cases} \tag{1.1.6}$$

where Δ^2 is the biharmonic operator: $\Delta^2 \cdot = \partial_{1111} \cdot + 2\partial_{1122} \cdot + \partial_{2222} \cdot$ The condition $\gamma_0 \Delta u|_\Gamma = 0$ is "natural" that is in weak sense.