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SOLVING BOUNDARY VALUE PROBLEMS FOR THE MATRIX EQUATION $X^{(2)}(t) - AX(t) = F(t)$ *

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Abstract

In this paper we present a method for solving the matrix differential equation $X^{(2)}(t) - AX(t) = F(t)$, without increasing the dimension of the problem. By introducing the concept of co-square root of a matrix, existence and uniqueness conditions for solutions of boundary value problems related to the equation as well as explicit solutions of these solutions are given, even for the case where the matrix A has no square roots.

§1. Introduction

Second order matrix differential equations with constant coefficients appear in the theory of damped systems and vibrational systems[5,7]. Explicit formulas for solutions of the matrix differential equation

$$X^{(2)} - AX = 0 \tag{1.1}$$

have been obtained in [1], but such formulas are not helpful for solving Cauchy problems and boundary value problems related to the non-homogeneous matrix equation

$$X^{(2)}(t) - AX(t) = F(t). \tag{1.2}$$

In a recent paper [9] we studied the boundary value problem

$$\begin{aligned} &X^{(2)}(t) - AX(t) = 0, \quad 0 \le t \le a, \\ &E_1X(0) + E_2X^{(1)}(0) = 0, \quad F_1X(a) + F_2X^{(1)}(a) = 0 \end{aligned} \tag{1.3}$$

where $E_i, F_i, i = 1,2, A$ and $X(t)$ are $n \times n$ complex matrices. Under the invertibility hypothesis of matrix A, conditions for the existence of non-trivial solutions of (1.3) and explicit expressions of these solutions in terms of appropriate square roots of A are given.

In this paper we extend the concept of square root of a square matrix A. Hence we get an interesting expression for the general solution of equation (1.1) that allows us to obtain a generalized variation of the parameter method for the matrix equation (1.2). Finally, from the expression of the general solution of equation (1.2), existence and uniqueness conditions for solutions of the problem

$$\begin{aligned} &X^{(2)}(t) - AX(t) = F(t), \quad 0 \le t \le a, \\ &E_1X(0) + E_2X^{(1)}(0) = 0, \quad F_1X(a) + F_2X^{(1)}(a) = 0 \end{aligned} \tag{1.4}$$

as well as an explicit expression of these solutions are obtained.

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The paper is organized as follows. In Section 2 we introduce the concept of co-square root of a complex matrix A and the concept of a fundamental pair of co-square roots of a matrix. An easy characterization of this last concept and a method for obtaining fundamental pairs of co-square roots are presented. Section 3 is concerned with problems (1.2) and (1.4).

§2. Co-Square Roots of Matrices

We begin this section with the concept of co-square root of a matrix A, that is a generalization of the well-known concept of square root of a matrix. This concept will be used below to solve problems (1.2) and (1.4), and it is interesting because there are matrices without square roots[4].

In the following the set of all $n \times n$ complex matrices will be denoted by $\mathbb{C}_{n\times n}$. If A is a matrix in $\mathbb{C}_{n\times n}$, we represent by $\sigma(A)$ the set of eigenvalues of A. For a rectangular complex matrix N we will represent by N^+ the Penrose-Moore pseudoinverse of N. An account of properties and applications of this concept may be found in [10].

Definition 2.1. *Let A be a matrix in $\mathbb{C}_{n\times n}$. We say that a pair of matrices, $(X,T) \in \mathbb{C}_{n\times n} \times \mathbb{C}_{n\times n}$, is a co-square root of A, if $X \neq 0$ and*

$$XT^2 - AX = 0. \tag{2.1}$$

Example 1. If $A \in \mathbb{C}_{n\times n}$ and B is a square root of A, and I denotes the identity matrix in $\mathbb{C}_{n\times n}$, then (I,B) is a co-square root of A.

Example 2. Let z be an eigenvalue of A, and let w be a complex number such that $w^2 = z$; then the kernel of $(w^2I - A)$ is nontrivial. Thus, for any nonzero matrix $X \in \mathbb{C}_{n\times n}$ such that $(w^2I - A)X = 0$, the pair (X, wI) is a co-square root of A.

Definition 2.2. *Let $A \in \mathbb{C}_{n\times n}$, and let (X_i, T_i) for $i = 1,2$ be co-square roots of A. We say that the pair $\{(X_1,T_1),(X_2,T_2)\}$ is a fundamental system of co-square roots of A, if the block partitioned matrix*

$$V = \begin{bmatrix} X_1 & X_2 \\ X_1T_1 & X_2T_2 \end{bmatrix} \tag{2.2}$$

in invertible in $\mathbb{C}_{2n\times 2n}$.

Example 3. Let $A \in \mathbb{C}_{n\times n}$, and let us suppose that T_1, T_2 is a pair of square roots of A; then the pair $\{(I,T_1),(I,T_2)\}$ is a fundamental system of co-square roots of A, if and only if the matrix $T_2 - T_1$ is invertible; see lemma 1 of [9] for details.

The next result provides a characterization for the existence of a fundamental system of co-square roots of a matrix A, and it shows that for a very general class of matrices $A \in \mathbb{C}_{n\times n}$ the construction of a fundamental pair of co-square roots is available.

Theorem 1. *Let $A \in \mathbb{C}_{n\times n}$ and let $C_L = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$. Then A admits a fundamental system of co-square roots, if and only if the matrix C_L is similar to a block diagonal matrix J of the form $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, where $J_i \in \mathbb{C}_{n\times n}$ for $i = 1,2$. In this case, if $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and $PJ = C_LP$, then the pair $\{(P_{11},J_1),(P_{12},J_2)\}$ defines a fundamental system of co-square roots of A.*