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A GENERAL NUMERICAL METHOD FOR SOLVING RIEMANN PROBLEMS*[1)]

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Abstract

In this paper a general numerical method for solving Riemann problems is discussed. It can be used to solve the Riemann problems of various hyperbolic systems of differential equations with two independent variables. The problem of reflection of discontinuities from external boundaries can also be solved by this general numerical method.

§1. Introduction

It is well-known that the Riemann problem plays a very important role in the theory of hyperbolic differential equations. Solving Riemann problems is the foundation of many numerical methods in this field. In the singularity-separating method[1],[2], in order to deal with interactions between discontinuities and reflection of discontinuities from boundaries accurately, it is also necessary to solve Riemann problems or some similar problems. In this paper a general numerical method is discussed. It can be used to solve the Riemann problem of various hyperbolic systems of differential equations with two independent variables. It is natural that users must give some necessary information when they use it. For example, the number of equations, the number of the distinct characteristic values, the multiplicity of every characteristic value, the physical quantities on the left and the right sides, the relations between physical quantities on the two sides of every kind of discontinuity, the entropy conditions, the relations among the physical quantities in every kind of central wave, the values of some parameters in equations and so on. If the above information is given, the solution of the Riemann problem, the types and the velocities of the discontinuity lines and weak discontinuity lines and the physical quantities between them, can be determined by our numerical method.

§2. Determining Equations for Riemann Problems

To solve a Riemann problem means to find out the solution of a system of quasilinear hyperbolic differential equations

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \tag{1}$$

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with the following kind of initial value

$$U(x,t_0)=\begin{cases} U_l, & x<0, \\ U_r, & x>0. \end{cases} \tag{2}$$

Here $U(x,t)$ is an n-dimensional vector, $A(U)$ is an $n\times n$ matrix with n real characteristic values and U_l and U_r are n-dimensional constant vectors.

Suppose among the n characteristic values there are l distinct ones, which are $\lambda^{(1)}$, $\lambda^{(2)}$, $\cdots$, $\lambda^{(l)}$ in the decreasing order, i.e.,

$$\lambda^{(1)}>\lambda^{(2)}>\cdots>\lambda^{(l)}.$$

In this case the structure of solution for a Riemann problem is as follows (see Fig.1)[3]: it consists of $l-1$ new constant-value subregions and l discontinuity lines—l boundary lines of the $l-1$ subregions. Here a discontinuity line means a real discontinuity or a central wave for the case that the state equation is convex, which we assume in this paper.

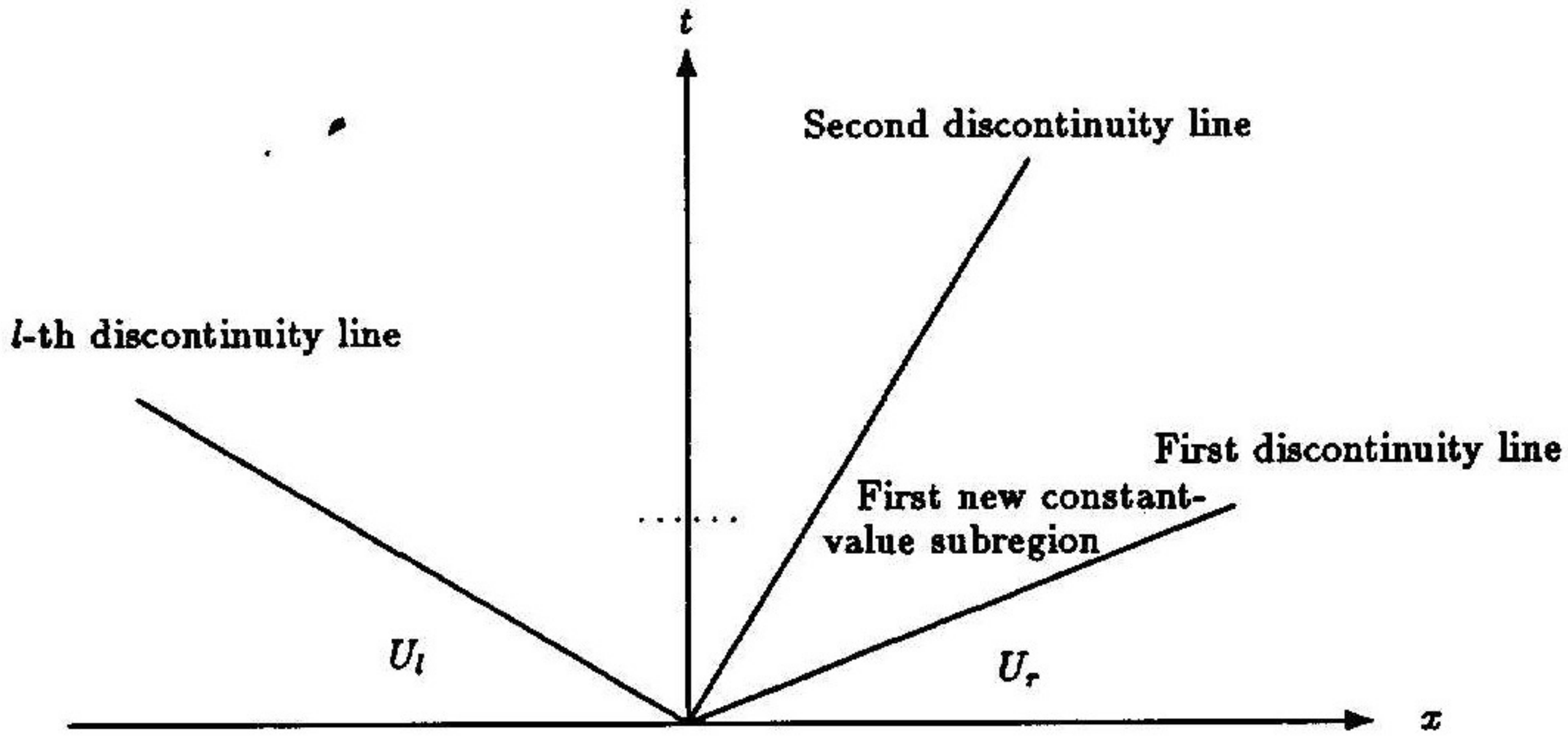

Fig. 1

If $\lambda^{(i)}$ is a $k^{(i)}$-fold characteristic value, there are $n+1-k^{(i)}$ discontinuity relations on the i-th kind of discontinuity line, where discontinuity relations means jump conditions on discontinuities or central wave relations. In practice, the jump conditions are nonlinear equations and the central wave relations, generally speaking, are ordinary differential equations. These discontinuity relations together form a system of equations which will be used to determine the solution of Riemann problems.

Because

$$\sum_{i=1}^{l}\left(n+1-k^{(i)}\right)=l(n+1)-\sum_{i=1}^{l}k^{(i)}=l(n+1)-n=(l-1)n+l,$$

the total number of equations in the system is $(l-1)n+l$. The unknown quantities are l velocities of discontinuity lines and $(l-1)n$ physical quantities in the $l-1$ subregions.