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A COMPLETELY EXPONENTIALLY FITTED DIFFERENCE SCHEME FOR A SINGULAR PERTURBATION PROBLEM*

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Abstract

A completely exponentially fitted difference scheme is considered for the singular perturbation problem: $\varepsilon U'' + a(x)U' - b(x)U = f(x)$ for $0 < x < 1$, with $U(0)$, and $U(1)$ given, $\varepsilon \in (0,1]$ and $a(x) > \alpha > 0, b(x) \geq 0$. It is proven that the scheme is uniformly second-order accurate.

§1. Introduction

The singular perturbation problem

$$\begin{aligned} &L_\varepsilon U_\varepsilon(x) \equiv \varepsilon U'' + a(x)U' - b(x)U = f(x), \qquad 0 < x < 1 \\ &U(0) = \beta_0,\ U(1) = \beta_1, \end{aligned} \tag{1.1}$$

where ε is a parameter in $(0, 1]$, $a(x) > \alpha > 0, b(x) \geq 0, x \in [0,1]$, is one of those main problems computational mathematicians are trying to solve. In dealing with this problem, some mathematicians are most interested in those numerical methods and schemes which are valid for the small parameter ε. In 1969, Il' in [1] designed an exponentially fitted finite difference scheme in the case $b(x) \equiv 0$, and showed that its solution converges uniformly in ε, with order one, to the solution of (1.1). Kellogg and Tsan [2] (1978), Miller [3] (1979) and Emelyanov [4] (1978) independently extended this result to the case $b(x) \geq 0$. Wu Qiguang [5] (1985) studied a class of weighted exponentially fitted difference schemes and proved that these schemes are uniformly convergent with order one. We have reason to say that satisfactory results have been obtained from the researches on the schemes which converge uniformly in ε with order one. In recent years, some mathematicians began to study higher order schemes. Hegarty, Miller and O'Riordan [6], Berger, Solomon and Ciment [7] proved separately that the completely exponentially fitted difference scheme (also called exponential box scheme) is convergent uniformly in ε with order two in the case $b(x) = 0$, which was derived by El-Mistikawy and Werle [8] in 1978. However, they could only conjecture through their numerical experiments that the same is true in the case $b(x) \geq 0$. In this paper, the scheme will be written in the form of completely exponentially fitting factor by introducing a dominant algebraic quantity. We will prove theoretically the conjecture of Berger etal. and complete soundly El-Mistikawy and Werle's scheme.

For the sake of convenience, we assume $a(x), b(x)$ and $f(x)$ are smooth enough, and throughout this paper those positive constants which are independent of ε, h and x_j will be generically denoted by C.

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§2. Asymptotic Properties of the Solution

In order to give bounds for the discretisation error for the scheme which are uniform in ε and h, we need the asymptotic expansion of the solution of (1.1). We first consider the following problem

$$\begin{aligned} L_\varepsilon U_\varepsilon(x) &= f_\varepsilon(x), \qquad 0 < x < 1, \\ U_\varepsilon(0) &= \beta_0(\varepsilon), \; U_\varepsilon(1) = \beta_1(\varepsilon). \end{aligned} \tag{2.1}$$

where $|\beta_0(\varepsilon)| \le C, |\beta_1(\varepsilon)| \le C, |f_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i-1}\exp(-\alpha x/\varepsilon)\}, i \ge 0$.

Noticing the Uniform boundness of $\beta_0(\varepsilon)$ and $\beta_1(\varepsilon)$, we have

Lemma 2.1 (Kellogg and Tsan [3]). *Problem* (2.1) *has a unique solution satisfying*

$$|U_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i}\exp(-\alpha x/\varepsilon)\}, \quad i \ge 0. \tag{2.2}$$

Lemma 2.1 *leads to the following lemma immediatly.*

Lemma 2.2 *The solution of problem* (2.1) *can be written in the form*

$$U_\varepsilon(x) = \delta V_\varepsilon(x) + Z_\varepsilon(x) \tag{2.3}$$

where $|\delta| \le C$ *is a constant,* $V_\varepsilon(x) = \exp(-\alpha(0)x/\varepsilon)$, $\alpha_\varepsilon(x) = \sqrt{a(x)^2 + 4\varepsilon b(x)}$ *and* $|Z_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i+1}\exp(-\alpha x/\varepsilon)\}, i \ge 0$.

As for problem (1.1), *we have*

Lemma 2.3 (Smith [9]). *The Solution of Problem* (1.1) *can be written in the form*

$$U_\varepsilon(x) = A_\varepsilon(x) + C_0 B(x)\exp(-\frac{1}{\varepsilon}\int_0^x a(x)ds) + R_\varepsilon(X) \tag{2.4}$$

where $|C_0| \le C$ is a constant, $|A_\varepsilon^{(i)}(x)| \le C, i \ge 0, B(x) = \frac{1}{a(x)}\exp\Big(-\int_0^x \frac{b(x)}{a(x)}ds\Big)$ and $R_\varepsilon(x)$ satisfies

$$\begin{aligned} L_\varepsilon R_\varepsilon(x) &= F_\varepsilon(x), \qquad 0 < x < 1, \\ R_\varepsilon(0) &= 0, \quad R_\varepsilon(1) = \beta(\varepsilon), \end{aligned}$$

where

$$|\beta(\varepsilon)| \le C, |F_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i}\exp(-\alpha x/\varepsilon)\}, i \ge 0.$$

Summing up the above lemmas, we conclude

Theorem 2.4. *Problem* (1.1) *has a unique solution which can be expressed as*

$$U_\varepsilon(x) = A_\varepsilon(x) + C_0 G_\varepsilon(x) + \varepsilon R_\varepsilon(x)$$

where

$$G_\varepsilon(x) = W_\varepsilon(x)E_\varepsilon(x), \quad R_\varepsilon(x) = \delta V_\varepsilon(x) + Z_\varepsilon(x)$$

and $|C_0|, |\delta| \le C$ are constants, $|A_\varepsilon^{(i)}(x)| \le C, i \ge 0$,

$$W_\varepsilon(x) = \frac{1}{a(x)}\exp\Big[-\int_0^x \Big(\frac{b(s)}{a(s)} + \frac{a(s) - \alpha(s)}{\varepsilon}\Big)ds\Big], \quad E_\varepsilon(x) = \exp\Big(-\frac{1}{\varepsilon}\int_0^x \alpha(s)ds\Big),$$

$$V_\varepsilon(x) = \exp(-\alpha(0)x/\varepsilon), \quad |Z_\varepsilon^{(i)}(x)| \le C\{1 + \varepsilon^{-i+1}\exp(-\alpha x/\varepsilon), \quad i \ge 0.$$

It is clear that $|W_\varepsilon^{(i)}(x)| \le C, i \ge 0$. For the sake of convenience, the subscripts ε in the symbols of operators and functions will be omitted.