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CONSTRUCTION AND ANALYSIS OF A NEW ENERGY –ORTHOGONAL UNCONVENTIONAL PLATE ELEMENT*

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Abstract

The paper describes an interpolation procedure of formulating shape functions for a new energy–orthogonal plate element. Sample problems using the new element show satisfactory numerical results.

§1. Introduction

Bergan et al.[1,2] have recently proposed the "free formulation" scheme of unconventional finite element methods. The element stiffness matrix consists of two separate parts: $K = K_{rc} + K_h$, where K_{rc} corresponds to constant strain modes of shape functions and is independent of any form of high order modes, while K_h is determined by high order modes based on a conventional energy consideration[3]. The TRUNC element developed by Argyris et al.[3] is an example of Bergan's free formulation scheme, which is proved to be convergent for arbitrary mesh partitions[4]. Reference [5] provides a mathematical explanation of the free formulation scheme. It is observed that the scheme actually leads to a nonconforming element method with a specific form of interpolation of shape functions. Reference [6] gives a detailed mathematical analysis for Bergan's energy-orthogonal element based on the free formulation [2]. Its convergence together with error estimates are derived and a modification of Bergan's element with better convergence properties is proposed.

Bergan's free formulation scheme has been stated in [1, 2] by mechanical considerations. The derivation of the stiffness matrix K_{rc} corresponding to constant strain modes, however, appears somewhat difficult of access. While the analysis in [5] shows that the matrix K_{rc} is identical with the matrix resulting from the constant strain modes of Zienkiewicz's incompatible cubic element, the reason for choice of this particular matrix as K_{rc} regardless of any form of high order modes is still not clear at least from a view-point of mathematics.

The purpose of this paper is to present a modified scheme of free formulation in accordance with a simple convergence requirement of nonconforming finite elements. The element stiffness matrix formulated by this modified scheme is again consisting of two separate parts, one corresponds to constant strain modes and the other to high order modes of shape functions. However, the stiffness matrix K_{rc} now is simply derived from the convergence requirement. It seems a more direct way of derivation of K_{rc} than that in Bergan's scheme. The treatment of high order modes leaves the same as before, using the conventional method. Starting from the shape function space of Bergan's energy-orthogonal element, the modified scheme provides a new energy-orthogonal element. Numerical experiments show that this new element gives more accurate results than Bergan's. The convergence proof as

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well as the error estimates are derived. Along the line of this paper a general nine parameter unconventional element, not necessarily energy-orthogonal, may be constructed, which will be analyzed in another paper.

§2. Formulation of Shape Functions and the Element Stiffness Matrix

Given a triangle K with the vertices $p_i = (x_i, y_i)$, the area Δ and the diameter $h_K \le h$, we denote by λ_i the area coordinates for the triangle K and put

$$\xi_1 = x_2 - x_3,\quad \xi_2 = x_3 - x_1,\quad \xi_3 = x_1 - x_2,\quad \eta_1 = y_2 - y_3,\quad \eta_2 = y_3 - y_1,\quad \eta_3 = y_1 - y_2,$$

$$F_i^2 = \xi_i^2 + \eta_i^2,\quad t_i = F_i^2/\Delta,\quad r_i = (\xi_j\xi_k + \eta_j\eta_k)/\Delta,\quad e_{ij} = \frac{r_i}{t_j},$$

$$i, j, k = 1, 2, 3,\quad j, k \ne i,\quad j \ne k.$$

The nodal parameters are the function values and the two first derivatives of w at the vertices p_i, which are denoted by

$$w = (w_1, w_{1x}, w_{1y}, w_2, w_{2x}, w_{2y}, w_3, w_{3x}, w_{3y})^T. \tag{2.1}$$

The space of shape functions under consideration is of the form

$$P(K) = \text{span}\,\{\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, N_7, N_8, N_9\}, \tag{2.2}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_3\lambda_1$ are constant strain modes and N_7, N_8, N_9 are high order modes. Every function $w \in P(K)$ may be written in the form

$$w = \bar{w} + w' \tag{2.3}$$

with

$$\bar{w} = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1,$$

$$w' = b_7N_7 + b_8N_8 + b_9N_9,$$

representing a constant strain term and a high order term, respectively.

In order to determine $\bar{w}$ we use an interpolation technique like the treatment of Morley's element[7]. Let the function value of $\bar{w}$ at the vertex p_i be identical with that of w at the same vertex, and the normal derivative of $\bar{w}$ at the middle point of one side be identical with the average of normal derivatives of w at the two end points of the side, i.e.

$$\bar{w}(p_i) = w_i,$$

$$\frac{\partial \bar{w}}{\partial n_i}(p_{jk}) = \frac{1}{2}\Big[\Big(\frac{\partial w}{\partial n_i}\Big)_j + \Big(\frac{\partial w}{\partial n_i}\Big)_k\Big] = \frac{1}{2F_i}\Big[-(w_{jx} + w_{kx})\eta_i + (w_{jy} + w_{ky})\xi_i\Big], i = 1, 2, 3, \tag{2.4}$$

where n_i denotes the unit outward normal vector of the side p_jp_k, opposite to the vertex p_i, and p_{jk} is the middle point of p_ip_k.

The interpolation conditions (2.4) uniquely determine the six coefficients $a_i, i = 1, \cdots, 6$, of $\bar{w}$ as follows: