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SYMPLECTIC COLLOCATION SCHEMES FOR HAMILTONIAN SYSTEMS *

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Abstract

The symplectic collocation schemes, which are based on the framework established by Feng Kang [1], are proposed for numerical solution of Hamiltonian systems. The sufficient and necessary conditions for various collocation schemes to be symplectic are obtained. Some examples of symplectic collocation schemes are also given.

§1. Introduction

Physics is characterized by conservation laws and symmetry [2]. This point of view suggests that the corresponding numerical method to be designed should preserve these invariants as much as possible, so that the corresponding numerical result could exactly reflect the essence of a real physical process. The Hamiltonian system describes a negligible dissipative physical process which plays an important role in the dynamical system. In recent years, numerical methods for the Hamiltonian system have received extensive attention. Due to the special structures and properties of this system (e.g., the stable phase flow of the system is noncontractible and preserves symplectic structures), the mumerical are naturally expected to be designed in such a way that these characters could be taken into full account.

For the consideration mentioned above, Feng Kang [1] proposed a new approach for the Hamiltonian system, the symplectic numerical method, which appears to be an active and interesting subject. Moreover, Feng and his group have systematically studied the symplectic difference schemes in recent years [1, 4, 5].

The spline function is one of the most useful mathematical tools in numerical analysis, and it is connected inherently with the generalized Hamiltonian system [3]. In this paper, we study the spline collocation method for the Hamiltonian system within the framework established by Feng [1]. The sufficient and necessary conditions for various operator spline collocation schemes to be symplectic are obtained.

§2. The Symplectic Collocation Scheme

Consider the canonical system of equations

$$\frac{dz}{dt} = K^{-1}H_z(z) \tag{2.1}$$

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where $z = (z_1.\cdots, z_{2n})^T \in R^{2n}$, K is a given nonsingular anti-symmetric matrix of order $2n$, and $H(z)$ a Gateaux differentiable functional called Hamiltonian of the system.

The 2-form defined on R^{2n} by

$$\omega_k = \sum_{i<j} k_{ij} dz_i \wedge dz_j \tag{2.2}$$

gives a natural symplectic structure on R^{2n}. Let g^t be the stable phase flow of system (2.1). Then g^t is noncontiractible, or generally speaking, g^t preserves the symplectic structure, i.e.,

$$(g^t)^* \omega_k = \omega_k. \tag{2.3}$$

Let $S(t)$ be a collocation solution for canonical equation (2.1) with knots $\{t_i\} : t_0 < t_1 < t_2 < \cdots$. A one-step collocation scheme (e.g., $S(t)$ is a spline function of degree less than three) is said to be symplectic if the transition $S(t_i) \to S(t_{i+1})$ from the i-th time step $S(t_i)$ to the next $(i+1)$-th time step $S(t_{i+1})$ is K-canonical for all i. (A transition is called a K-canonical one if it preserves symplectic structure (2.2)).

§3. Collocation Scheme Using Operator Spline of Order Two

Let $S(t)$ be a second order operator spline with knots $\{t_i\}, S(t) \in C^0$,

$$S(t) = c_i \varphi_1(t) + d_i \varphi_2(t), \quad t_i \le t \le t_{i+1}, \tag{3.1}$$

where $\varphi_1(t), \varphi_2(t)$ is a Tchebysheff system (i.e., φ_1, φ_2 satisfies the Haar condition) and c_i, d_i are constants. The collocation scheme for (2.1) is

$$\dot{S}(t_{i+1/2}) = K^{-1} H_z(S(t_{i+1/2})), \quad i = 0, 1, 2, \cdots \tag{3.2}$$

In the following discussion, we denote $S(t_i)$ by S_i, and $S(t_{i+1/2})$ by $S_{i+1/2}$, etc.

Theorem 3.1. *Collocation scheme* (3.2) *is symplectic if and only if the following equalities*

$$(\alpha_i' + \beta_i')(\alpha_i' - \beta_i')I = (\alpha_i - \beta_i)(\alpha_i + \beta_i)(K^{-1} H_{zz}(S_{i+1/2}))^2, \quad i = 0, 1, 2, \cdots \tag{3.3}$$

hold, where

$$\alpha_i = \phi(\sqrt{2}), \ \alpha_i' = d\phi(1/2)/dt, \ \beta_i = \psi(1/2), \quad \beta_i' = d\psi(1/2)/dt,$$

$$\phi(t) = [\varphi_2(t_{i+1})\varphi_1(t) - \varphi_1(t_{i+1})\varphi_2(t)]/\det \begin{pmatrix} t_i & t_{i+1} \\ \varphi_1 & \varphi_2 \end{pmatrix},$$

$$\psi(t) = [\varphi_1(t_i)\varphi_2(t) - \varphi_2(t_i)\varphi_1(t)]/\det \begin{pmatrix} t_i & t_{i+1} \\ \varphi_1 & \varphi_2 \end{pmatrix},$$

and $H_{zz}(S_{i+1/2})$ is the Hessian matrix of the function $H(z)$.

Proof. From (3.1),

$$S_i = c_i \varphi_1(t_i) + d_i \varphi_2(t_i),$$

$$S_{i+1} = c_i \varphi_1(t_{i+1}) + d_i \varphi_2(t_{i+1}),$$