Journal of Computational Mathematics, Vol.8, No.4, 1990, 371–380.

SYMPLECTIC DIFFERENCE SCHEMES FOR LINEAR HAMILTONIAN CANONICAL SYSTEMS*[1)]

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Abstract

In this paper, we present some results of a study, specifically within the framework of symplectic geometry, of difference schemes for numerical solution of the linear Hamiltonian systems. We generalize the Cayley transform with which we can get different types of symplectic schemes. These schemes are various generalizations of the Euler centered scheme. They preserve all the invariant first integrals of the linear Hamiltonian systems.

§1 Introduction

Recently, it becomes evident that the hamiltonian formalism plays a fundamental role in mathematical physics.One needs only to recall a few examples: classical mechanics, quantum mechanics, hydrodynamics of a perfect fluid, plasma physics, and accelerator physics.

The evolution of Hamiltonian systems has the important property of being symplectic, i.e., the sum of the areas of the canonical variable pairs, projected on any two-dimensional surface in a phase space, is time invariant. In numerically solving these equations it is necessary to replace them with finite difference equations which preserve this symplectic evolution property. In [1] the first author proposed a systematic study of symplectic difference schemes for hamiltonian systems from the viewpoint of symplectic geometry. We present here some developments for linear hamiltonian systems.

An outline of this paper is as follows: Section 2 is devoted to a review of well known facts concerning symplectic structures and hamiltonian mechanics. In Section 3 we review some properties of the symplectic matrix and the infinitesimal symplectic matrix. In Section 4 we review some linear symplectic difference schemes. Constructions of linear symplectic schemes based on the Padé approximation are described in §5. Generalized Cayley transform and its corresponding symplectic schemes and conservation laws are presented in §6.

§2 Some Facts from Hamiltonian Mechanics and Symplectic Geometry

In this section we will review some facts from Hamiltonian mechanics and symplectic geometry which are fundamental to what follows. Consider the following system of differential

*Received February 9, 1988.
1)The Project Supported by National Natural Science Foundation of China.

equations on R^{2n}

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i},$$
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad i = 1, 2, ..., n \tag{2.1}$$

where H(p,q) is some real valued smooth function on R^{2n}. We call (2.1) a canonical system of differential equations with Hamiltonian H. We denote $p_i = z_i, q_i = z_{i+n}, z = (z_1, ... z_{2n})'$, and $\frac{\partial H}{\partial z} = (\frac{\partial H}{\partial z_1}, ..., \frac{\partial H}{\partial z_{2n}})' \in R^{2n}$. Then (2.1) becomes

$$\frac{dz}{dt} = J^{-1}\frac{\partial H}{\partial z} \tag{2.2}$$

with

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad J' = -J = J^{-1} \tag{2.3}$$

where I_n is the identity matrix. The phase space R^{2n} is equipped with a standard symplectic structure defined by the "fundamental" differential 2-form

$$\omega = \sum_1^n dp_i \wedge dq_i.$$

Let g be a diffeomorphism of R^{2n}:

$$z = \begin{pmatrix} p \\ q \end{pmatrix} \to g(z) = \begin{bmatrix} g_1(z) \\ \vdots \\ g_{2n}(z) \end{bmatrix} = \begin{bmatrix} \hat{p}(p,q) \\ \hat{q}(p,q) \end{bmatrix}$$

g is called a symplectic transformation if g preserves the 2-form ω, i.e.,

$$\sum_1^n d\hat{p}_i \wedge d\hat{q}_i = \sum_1^n dp_i \wedge dq_i.$$

This is equivalent to the condition that

$$\left(\frac{\partial g}{\partial z}\right)' J \left(\frac{\partial g}{\partial z}\right) \equiv J$$

i.e. the Jacobian matrix $\frac{\partial g}{\partial z}$ is symplectic everywhere,

$$\frac{\partial g}{\partial z} = \begin{bmatrix} \frac{\partial \hat{p}}{\partial p} & \frac{\partial \hat{p}}{\partial q} \\ \frac{\partial \hat{q}}{\partial p} & \frac{\partial \hat{q}}{\partial q} \end{bmatrix}.$$