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APPROXIMATE SEVERAL ZEROES OF A CLASS OF PERIODICAL COMPLEX FUNCTIONS*

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Abstract

This paper discussed the number of zeroes of the complex function $F : C \to C$ defined by

$$F(Z) = \sum_{k=1}^{n}(a_k \cos(kZ) + b_k \sin(kZ)) + \alpha_0 + \alpha_1 \mathrm{Im}(Z) + \cdots + \alpha_m(\mathrm{Im}(Z))^m,$$

where $\mathrm{Im}(Z)$ is the imaginary part of $Z, |a_n|+|b_n| \neq 0$. Let $n_1 = \max\limits_{1\le k\le n}\{0, k|b_k \neq -ia_k\}$ and $n_2 = \max\limits_{1\le k\le n}\{0, k|b_k \neq ia_k\}$. We prove that if 0 is a regular value of F and $n_1 n_2 \neq 0$, then F has at least $n_1 + n_2$ zeroes in domain $(0, 2\pi] \times R$ and $n_1 + n_2$ of them can be located with the homotopy method simultaneously. Furtheromore, if $\alpha_1 = \cdots = \alpha_m = 0$ and $n_1 n_2 \neq 0$, then F has exactly $n_1 + n_2$ zeroes in domain $(0, 2\pi] \times R$.

§1. Introduction

Let C be the complex plane. We regard C as R^2 by identifying $Z = x + iy \in C, x, y \in R$ with $(x, y) \in R^2$. Define a complex function $F : C \to C$ by

$$F(Z) = T(Z) + f(Z), \tag{1.1}$$

where T is a triangular polynomial with degree n and f is a polynomial of $\mathrm{Im}(Z)$ with degree m. That is

$$T(Z) = \sum_{k=1}^{n}(a_k \cos(kZ) + b_k \sin(kZ)),$$

$$f(Z) = \alpha_0 + \alpha_1 \mathrm{Im}(Z) + \cdots + \alpha_m(\mathrm{Im}(Z))^m,$$

where a_k, b_k, α_j are all complex numbers and $\alpha_m \neq 0, |a_n| + |b_n| \neq 0$.

By the definition of F, F is a periodical function of Z with period 2π. So we need only to discuss the zero distribution of F in domain $(0, 2\pi] \times R$. Section 2 studies the number of zeroes of F and develops a method to calculate several zeroes of F. Section 3 gives some numerical examples.

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§2. Approximate the Zeroes

Let $\phi : R^p \to Q^q$ be a smooth mapping. Let $x \in R^p$ be a regular point if the Jacobian matrix of ϕ at x is of full rank. We call $y \in R^q$ a regular value of ϕ if $\phi^{-1}(y) + \{x \in R^p | \phi(x) = y\}$ contains only regular points of ϕ.

Lemma 1[1]. *Let $\phi : R^p \times R^q \to R^r$ be a smooth mapping. If 0 is a regular value of ϕ, then for almost all $d \in R^q$, 0 is a regular value of the mapping $\phi(\cdot, d) : R^p \to R^r$.*

Consider the function F of form (1.1). Since $\dfrac{\partial F}{\partial \alpha_0} = 1$, by Lemma 1, for almost all $\alpha_0 \in C$, 0 is a regular value of F. In this section, we always assume that 0 is a regular value of F.

Lemma 2[2]. *Let $H : R^n \times [0,1] \to R^n$ be a smooth mapping. Suppose 0 is a regular value of $H, H(\cdot, 0) : R^n \to R^n$ and $H(\cdot, 1) : R^n \to R^n$. Let (x^1, t^1) and (x^2, t^2) be two boundary points of a component of $H^{-1}(0)$.*

(a) If $t^1 = t^2$, then

$$\text{sgn det } \frac{\partial H}{\partial x}(x^1, t^1) = -\text{sgn det } \frac{\partial H}{\partial x}(x^2, t^2).$$

(b) If $t^1 \neq t^2$, then

$$\text{sgn det } \frac{\partial H}{\partial x}(x^1, t^1) = \text{sgn det } \frac{\partial H}{\partial x}(x^2, t^2),$$

where sgn is the sign funciton.

Let $F = T + f : C \to C$ be as in (1.1). T is a triangular polynomial with degree n. Define the auxiliary function $G : C \to C$ by

$$G(Z) = c(e^{in_1 Z} + e^{-in_2 Z}), \tag{2.1}$$

where c is a nonzero complex number. It is easy to know that G has exactly $n_1 + n_2$ zeroes in domain $(0, 2\pi] \times R$; they are

$$Z = \frac{2k+1}{n_1 + n_2}\pi, \quad k = 0, 1, \cdots, n_1 + n_2 - 1,$$

and 0 is a regular value of G.

Define homotopy $E : C \times [0,1] \times C \to C$ by

$$E(Z, t, \alpha) = tF(Z) + (1-t)G(Z) + t(1-t)\alpha. \tag{2.2}$$

Then, $E(\cdot, 0, \cdot) = G(\cdot)$ and $E(\cdot, 1, \cdot) = F(\cdot)$. Since 0 is a regular value of F and G, and

$$\frac{\partial E}{\partial \alpha} = t(1-t),$$

by Lemma 1, for almost all $\alpha \in C$, 0 is a regular value of $H(\cdot, \cdot) = E(\cdot, \cdot, \alpha) : C \times [0,1] \to C$. Fix $\alpha \in C$ such that 0 is a regular value of H. $H^{-1}(0) = \{(Z, t) \in C \times [0,1] | H(Z, t) = 0\}$ is a one-dimensional manifold. That is, $H^{-1}(0)$ consists only of simple smooth curves.