EXTRAPOLATION FOR THE APPROXIMATIONS TO THE SOLUTION OF A BOUNDARY INTEGRAL EQUATION ON POLYGONAL DOMAINS

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Abstract

In this paper, we consider a boundary integral equation of second kind rising from potential theory. The equation may be solved numerically by Galerkin's method using piecewise constant functions. Because of the singularities produced by the corners, we have to grade the mesh near the corner. In general, Chandler obtained the order 2 superconvergence of the iterated Galerkin solution in the uniform norm. It is proved in this paper that the Richardson extrapolation increases the accuracy from order 2 to order 4.

1. Introduction

Let $\Gamma \subset R^2$ be a simple closed polygon with corner points $x_0, x_1, x_2 \dots x_m = x_0$ For each $i, X_i \in (-1,1)$ is defined by requiring $(1-X_i)\pi$ to be the angle $x_{i-1}\widehat{x_i}x_{i+1}$. Let us consider the boundary integral equation of second kind

$$u_0(x) + Vu_0(x) = f(x), \qquad x \in \Gamma \tag{1.1}$$

where

$$Vu_0(x) = \int_\Gamma k(x,y)u_0(y)\ dy$$

with

$$k(x,y) = \frac{1}{\pi}\frac{\partial}{\partial n_y}\ln|x-y|$$

Assume that Γ is parametrized by arc length s , with $s = s_i$ corresponding to the point x_i. We do not distinguish s from the point length s around Γ, and use the expressions such as $u(s), k(s,\sigma)$, etc. Let $s_{i+1/2} = (s_i + s_{i+1})/2, = s_{i-1/2} = (s_i + s_{i-1})/2, \Gamma_{2i} = [s_{i-1/2}, s_i]$ and $\Gamma_{2i+1} = [s_i, s_{i+1/2}]$. Each function u on Γ may be identified with the vector $(u_2, \dots, u_{2m+1}), u_k := u|_{\Gamma_k}$. Then (1.1) is equivalent to the $2m \times 2n$ system of equations

$$(I+T)u_0 = f \tag{1.2}$$

with $f = (f_2, \dots, f_{2m+1}), u_0 = (u_{0,2}, u_{0,3}, \dots\dots, u_{0,2m+1})$ and where the matrix operator T is defined by

$$T_{k,l}u_l(s) = \int_{\Gamma_l} k(s,\sigma)u_l(\sigma)\, d\sigma, \quad s \in \Gamma_k,$$

$$(Tu)_k = \sum_{l=2}^{2m+1} T_{k,l}u_l.$$

When

$$\begin{aligned} &\{k,l\} = \{2i, 2i+1\}, s \in \Gamma_k, \sigma \in \Gamma_l \\ &k(s,\sigma) = (\sin X_i\pi/\pi)\cdot(s-s_i)/[(s-s_i)^2 + (s_i-\sigma)^2 \\ &+2(s-s_i)(s_i-\sigma)\cos X_i\pi]. \end{aligned} \tag{1.3}$$

When

$$k = l\ ,\ s \in \Gamma_k\ ,\ \sigma \in \Gamma_k,$$
$$k(s,\sigma) = 0.$$

T may be separated by writing $T = R + K$, where

$$\begin{aligned} R_{k,l} &= T_{k,l}, \qquad \{k,l\} = \{2i, 2i+1\} \quad \text{for some } i \\ &= 0, \qquad \text{otherwise} \end{aligned}$$

and the kernels of the components of K are smooth. For each i let R_i denote the 2×2 system of operators :

$$R_i = \begin{bmatrix} 0 & R_{2i,2i+1} \\ R_{2i+1,2i} & 0 \end{bmatrix}.$$

then $R = \mathrm{diag}[R_1, R_2, \dots, R_m]$.

For any $\alpha_i > 0$ and integer $k \geq 0$, define the norm

$$\|u\|_{k,\alpha_i} = \max_{m\leq k} \sup\left\{\left|[s-s_i]^{m-\alpha_i}D^m u(s)\right|:\ s\ \in \Gamma_{2i}\cup\Gamma_{2i+1}\backslash\{s_i\}\right\}$$

with $[s]^\beta = |s|^{\max(\beta,0)}$, and the space

$$C^k_{\alpha_i}(\Gamma_{2i}\cup\Gamma_{2i+1}) = \left\{u|u \in C^k(\Gamma_{2i}\cup\Gamma_{2i+1}\backslash\{s_i\}), \|u\|_{k,\alpha_i} < \infty\right\}.$$

For any vector $\alpha = (\alpha_1, \alpha_2, \dots\dots, \alpha_m), \alpha_i > 0$, define the Banach space

$$C^k_\alpha(\Gamma) = \left\{u|u \in C^k(s_{i-1}, s_i), \forall_i = 1, 2, \dots m, \|u\|_{k.\alpha} < \infty\right\}$$

with the norm

$$\|u\|_{k,\alpha} = \max_i\left\{\|u|_{\Gamma_{2i}\cup\Gamma_{2i+1}}\|_{k,\alpha_i}\right\}.$$