A REMARK ON MINIMAL SUPPORT FOR BIVARIATE SPLINES*

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§1. Introduction

This paper is concerned with the minimality of the support for bivariate splines, which was first investigated by de Boor and Höllig in [1]. Now let us introduce the problem briefly.

Let $\pi^{\rho}_{k,\Delta}$ be the space of bivariate pp (piecewise polynomial) functions in C^{ρ}, of degree $\le k$, on the mesh Δ obtained from a uniformly unit square mesh by drawing all upward sloping diagonals. It is well-known that whenever $\rho > [\frac{2k-2}{3}]$, there is no element in space $\pi^{\rho}_{k,\Delta}$ with compact support and this kind of spaces has bad approximating properties. So we always suppose $\rho \le [\frac{2k-2}{3}]$. In this case there exist compact support elements in space $\pi^{\rho}_{k,\Delta}$. Naturally, the problem of searching for the minimal support elements in this kind of spaces is raised. The very useful cases are $\rho = [\frac{2k-2}{3}]$. For different values of k mod 3, we obtain three different kinds of spaces $\pi^{\rho}_{k,\Delta}$. They will be denoted respectively by $S_{o,\mu} = \pi^{2\mu-3}_{3\mu-3,\Delta}$, $S_{1,\mu} = \pi^{2\mu-2}_{3\mu-2,\Delta}$ and $S_{2,\mu} = \pi^{2\mu-2}_{3\mu-1,\Delta}$, where μ is a positive integer.

In order to investigate $S_{o,\mu}$ and $S_{1,\mu}$, de Boor and Höllig introduced the following concepts.

A function f is said to have minimal support in space S if $f \in S$ and the only $g \in S$ having support strictly inside supp f is $g = 0$. And a function f is said to have unique minimal support in S if $f \in S$ and any $g \in S$ having support in supp f is a multiple of f (see[1]).

They determined all minimal support elements in $S_{0,\mu}$ and $S_{1,\mu}$. But when searching for minimal support elements in $S_{2,\mu}$, de Boor and Höllig noticed that if stuck to the definition of minimal support elements given earlier, one could not find a basis for $S_{2,\mu}$ consisting only of minimal support elements. As an example, they showed that the translates of four functions, whose supports are drawn in Figure 1, provide a suitable basis for $S_{2,1}$, but the notable fact is that the fourth element does not have minimal support. Without any definition or further illustration they still called them four "minimal support" elements and asserted at the end of their paper that for odd μ, the four "minimal support" elements in

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$S_{2,\mu}$ can be obtained from the preceding case by convolution with $M_{1,1,1}$ (for the definition of $M_{1,1,1}$ refer to [1]).

Since their "minimal support" in $S_{2,\mu}$ has an uncertain meaning, we cannot draw any clear conclusions from their discussion. We have to redefine the concept of minimal support for this case in order that we might discuss the problem precisely. That is what we want to do in this paper, which is arranged as follows: In Section 2, we calculate the dimension of the subspace of $S_{2,\mu}$, consisting of all the elements having their supports in a certain closed convex domain. In Section 3, the minimal support system in $S_{2,\mu}$ is defined and such a system in $S_{2,3}$ is given as well. We also show that the system consisting of the four "minimal support" elements in $S_{2,3}$, which were obtained and termed by de Boor and Höllig, is not a minimal support system.

§2. A local dimension theorem

Let $\Xi = (\xi_i)_1^n$ be a sequence in R^S. The truncated power (or cone spline) $C_\Xi(x)$ is defined as the following distribution on R^S

$$C_\Xi : \phi \to \int_{R_+^s} \phi\Big(\sum_{i=1}^n t(i)\xi_i\Big)dt,$$

where $\phi(x)$ is any test function on R^s and $R_+^s\{x \in R^s; x(i) \geq 0, i = \overline{1,s}\}$.

And the box spline $M_\Xi(x)$ is defined as follows:

$$\int_{R^S} \phi(x)M_\Xi(x)dx = \int_{[o,1]^n} \phi\Big(\sum_{i=1}^n t(i)\xi_i\Big)dt, \quad \phi \in C_0^\infty(R^s).$$

It is well-known that (see[2]) for $W \subset \Xi$

$$D_W C_\Xi = C_{\Xi\backslash W}.$$

In this paper, we only consider the case of R^2 and Ξ will be taken as $\Xi = (d_1 : r, d_2 : s, d_3 : t)$, where $d_1 = e_1 = (1,0), d_2 = e_2 = (0,1)$ and $d_3 = e_1 + e_2 = (1,1)$. For simplicity, we shall write $C_{r,s,t}, M_{r,s,t}$ instead of C_Ξ, M_Ξ respectively.

Lemma 2.1[3]. *Let $k = r + s + t - 2$; then*

$$C_{r,s,t}(x) = \begin{cases} p(x), & x \in V_1 = \{x; x \in R_+^2, x(1) - x(2) \leq 0\}; \\ q(x), & x \in V_2 = \{x; x \in R_+^2, x(1) - x(2) > 0\}, \end{cases}$$

where

$$p(x) = \frac{\binom{s+t-2}{t-1}}{k!}\sum_{j=0}^{s-1}(-1)^{s-1-j}\frac{\binom{s-1}{j}\binom{k}{j}}{\binom{s+t-2}{j}}x(1)^{k-j}x(2)^j,$$

$$q(x) = \frac{\binom{r+t-2}{t-1}}{k!}\sum_{j=0}^{r-1}(-1)^{r-1-j}\frac{\binom{r-1}{j}\binom{k}{j}}{\binom{r+t-2}{j}}x(1)^j x(2)^{k-j}.$$