HOW TO RECOVER THE CONVERGENT RATE FOR RICHARDSON EXTRAPOLATION ON BOUNDED DOMAINS*

LIN QUN (林 群) XIE RUI-FENG (谢锐锋)
(Institute of Systems Science, Academia Sinica, Beijing, China)

Abstract

We are interested in solving elliptic problems on bounded convex domains by higher order methods using the Richardson extrapolation. The theoretical basis for the application of the Richardson extrapolation is the asymptotic error expansion with a remainder of higher order. Such an expansion has been derived by the method of finite difference, where, in the neighborhood of the boundary one must reject the elementary difference analogs and adopt complex ones. This plight can be changed if we turn to the method of finite elements, where no additional boundary approximation is needed but an easy triangulation is chosen, i.e. the higher order boundary approximation is replaced by a chosen triangulation. Specifically, a global error expansion with a remainder of fourth order can be derived by the linear finite element discretization over a chosen triangulation, which is obtained by decomposing the domain first and then subdividing each subdomain almost uniformly. A fourth order method can thus be constructed by the simplest linear finite element approximation over the chosen triangulation using the Richardson extrapolation.

§ 1. Problem and Result

The Richardson extrapolation to the limit is a common way of increasing the accuracy of low order finite difference schemes applied to ordinary differential equations[23]. For elliptic equations, for example, the two-dimensional model problem

$$-\Delta u=f \quad \text{in } \Omega,\ u=0 \quad \text{on } \partial\Omega \tag{1}$$

on a curved domain Ω, the elementary difference analogs do not, near the boundary, allow us to expand the approximation error in powers of the mesh size h. Therefore, near the boundary we must reject the elementary difference analogs and adopt complex ones which usually lead to a large number of nonzero coefficients in the equations near the boundary. In so doing we shall succeed in obtaining an expression for the approximation error[7,18,19,25]

$$u^h(z)-u(z)=h^2e(z)+O(h^4) \tag{2}$$

at nodal points z.

What will happen to the method of finite elements? Can the additional higher order boundary approximation be avoided by choosing a proper triangulation?

Let us recall the L_2-error estimate for linear finite element approximation u^h,

$$u^h-u=O(h^2) \text{ in } L_2\text{-norm}.$$

It is hopeless, in contrast to the usual imagination, to prove the further error

---

* Received November 11, 1986.

expansion like

$$u^h - u = h^2 e + O(h^4) \quad \text{in } L_2\text{-norm}$$

since the combination of approximations on two triangulations (denoted by T^h and $T^{h/2}$, respectively) is still a piecewise linear function over $T^{h/2}$ which does not give an $O(h^4)$ approximation of u in L_2.

Let us turn to the pointwise estimate

$$u^h(z) - u(z) = O(h^2 |\log h|),$$

where the factor $\log h$ cannot be moved at the nodal points unless the surrounding meshes have strong symmetry[6,13] (forming a six-polygon at least). So, it seems that we cannot hope to find a united error expansion (2) for all nodal points of a general (regular) triangulation.

So, the problem is how to choose a triangulation such that there will exist a united error expansion (2) for all nodal points.

In [6], [14] and [15] a piecwise uniform triangulation, and in [17] a piecewise almost uniform triangulation are constructed in order to obtain the united error expansions like (2). These kinds of triangulation are easy to be constructed for the polygonal domains, for instance, by first choosing coarse triangles or quadrilaterals and then subdividing each triangle or quadrilateral almost uniformly.

An interior uniform triangulation has been used in [6, 16] for the curved domains. By an arbitrary arrangement of triangular meshes near the boundary we get only an interior error expansion with a reduced order $O(h^3|\log h|)$ for the remainder. It seems that unproper meshes near the boundary pollute the remainder even in the interior of Ω.

In [2], a transformed uniform triangulation was introduced in combining with a transformed bilinear element approximation. It is the purpose of this paper to describe how to recover usual linear elements from the transformed linear elements used in [2].

Define a triangulation T^h by first decomposing the domain and then subdividing each subdomain almost uniformly, for example, by the following possible process (see Fig. 1).

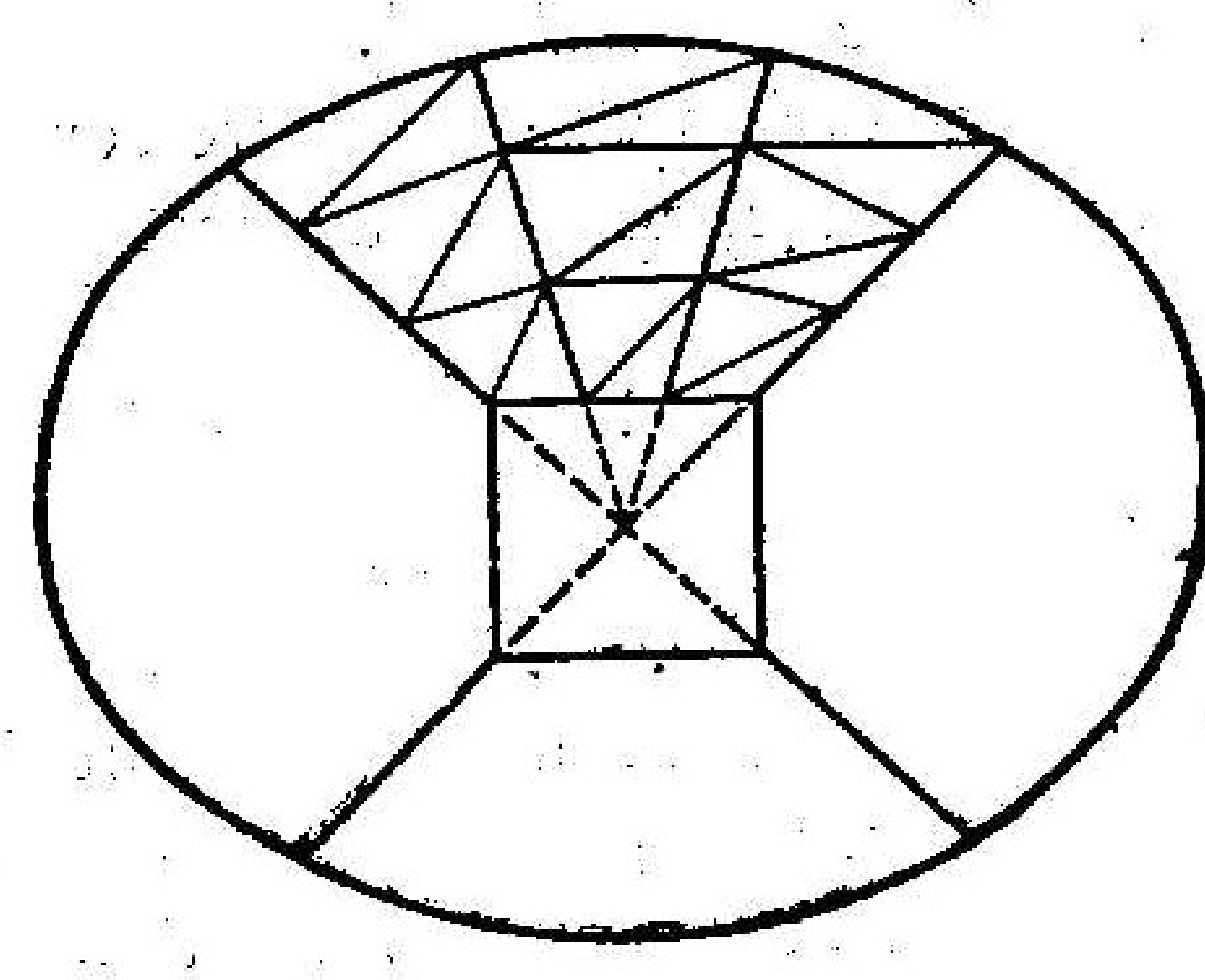

Fig. 1

Suppose, for simplicity, that Ω is a star domain with respect to a point O. Firstly, choose a square Ω_0 with its center at O and divide $\Omega \backslash \Omega_0$ into four pieces $\{\Omega_i,\ 1 \leqslant i \leqslant 4\}$ by four rays passing through O and the four vertices of Ω_0. Secondly, make n-equipartition along each edge of Ω_0 and draw $n-1$ rays through O and the $n-1$ equinodes. Linking the n-equinodes along each ray lying in Ω_i we obtain an almost uniform triangulation T_i^h over Ω_i. Finally, let T_0^h be a uniform triangulation over Ω_0. We obtain a piecewise almost uniform triangulation

$$T^h = \bigcup T_i^h \tag{3}$$

over $\Omega^h = \bigcup \{K \in T^h\}$.