FIXED POINT METHODS FOR THE COMPLEMENTARITY PROBLEM*[1)]

SUBRAMANIAN P. K.

(California State University, Los Angeles, USA)

Abstract

This paper is concerned with iterative procedures for the monotone complementarity problem. Our iterative methods consist of finding fixed points of appropriate continuous maps. In the case of the linear complementarity problem, it is shown that the problem is solvable if and only if the sequence of iterates is bounded in which case summability methods are used to find a solution of the problem. This procedure is then used to find a solution of the nonlinear complementarity problem satisfying certain regularity conditions for which the problem has a nonempty bounded solution set.

§ 1. Introduction

We are concerned in this paper with the complementarity problem, viz., that of finding a $z_0 \geqslant 0$ such that $F(z_0) \geqslant 0$ and such that $z_0^T F(z_0) = 0$. Here F is an operator from $\mathfrak{R}^n$ to $\mathfrak{R}^n$. In particular, we are concerned with the case when F is monotone, that is

$$(x-y)^T(F(x)-F(y)) \geqslant 0, \quad \forall x,\ y \in \mathfrak{R}^n.$$

The operator F is strongly monotone if there exists a positive real number λ such that

$$(x-y)^T(F(x)-F(y)) \geqslant \lambda \|x-y\|^2.$$

When F is an affine map, $F(x) = Mx+q$, we shall refer to the complementarity problem as the linear complementarity problem and write LCP(M, q) in this case. Otherwise we shall refer to it as the nonlinear complementarity problem and write NLCP(F). Clearly when F is affine and monotone, M is positive semidefinite.

In the case of LCP(M, q), when M is positive semidefinite, if the problem is feasible, that is there exists $x \geqslant 0$ such that $Mx+q \geqslant 0$, the problem is solvable [Eaves, 1971]. This is not the case for NLCP(F) ([Megiddo, 1977], [Garcia, 1977]). However, for $\varepsilon > 0$, if we consider the Tihonov regularization $F_\varepsilon := F + \varepsilon I$, then the corresponding problem NLCP(F_ε) has a unique solution since F_ε is strongly monotone [Karamardian, 1972]. When $\varepsilon \to 0$, x_ε converges to the least two-norm solution of NLCP(F), provided it is solvable [Brézis, 1973].

A solution of NLCP(F) is also a fixed point of the map

* Received July 28, 1986.

1) Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based on work sponsored by the National Science Foundation Grants DCR-8420963 and MCS-8102684, AFSOR Grant AFSOR-ISSA-85-0880.

$$x \mapsto (x-F(x))_+ := \max\{0, x-F(x)\}.$$

The principal aim of this paper is to consider iterative procedures to find such fixed points. We shall show that in the linear case the sequence of iterates is bounded if and only if LCP(M, q) is solvable. When this is the case, we use summability methods to obtain a solution of the problem. Although feasibility of the monotone NLCP(F) does not imply its solvability, it is a theorem of Mangasarian and McLinden [1985] that when a regularity condition such as the distribute Slater constraint qualification is satisfied, then, in this case, the solution set is bounded. We show how the iterative procedure for the linear case may be adapted to find a solution in this special case.

We briefly describe the notation used in this paper. We use $\Re^n$ for the space of real ordered n-tuples. All vectors are column vectors and we use the Euclidean norm throughout. Given a vector x, we denote its i^{th} component by x_i. We say $x \geqslant 0$ if $x_i \geqslant 0 \forall i$. The nonnegative orthant is denoted by $\Re^n_+$.

We use superscripts to distinguish between vectors, e.g. x^1, x^2 etc. For x, $y \in \Re^n$, x^T indicates the transpose of x, $x^T y$ their inner product. Occasionally, the superscript T will be suppressed. All matrices are indicated by upper case letters A, B, C, etc. The i^{th} row of A is denoted by A_i while its j^{th} column is denoted by A_j. The transpose of A is denoted by A^T.

Given NLCP(F), we define the feasible set and solution set by $S(F)$ and $\bar{S}(F)$ respectively, that is,

$$S(F) = \{x \in \Re^n_+ : F(x) \in \Re^n_+\},$$

$$\bar{S}(F) = \{x \in S(F) : x^T F(x) = 0\}.$$

In the case of LCP(M, q), we shall denote these sets by $S(M, q)$ and $\bar{S}(M, q)$ respectively. Finally the end of a proof is signified by ▌.

§ 2. Fixed Point Methods

We begin with the well known notion of a contraction mapping.

2.1. Definition. *Let P: $D \subseteq \Re^n \to \Re^n$. We say P is Lipschitzian with modulus $L > 0$ if*

$$\|P(x) - P(y)\| \leqslant L\|x-y\|, \quad \forall x, y \in D. \tag{2.1}$$

When $L \leqslant 1$ ($L < 1$) we say P is non-expansive (contractive).

The following theorem is classical; see e.g., [Ortega and Rheinboldt, 1970, page 120].

2.2. Theorem. *(Banach's contraction mapping principle). Let P: $D \subseteq \Re^n \to \Re^n$, D_0 a closed subset of D such that $PD_0 = \{P(x): x \in D_0\} \subseteq D_0$. If P is a contraction mapping on D_0 with modulus L, then P has a unique fixed point $\bar{x}$ in D_0. Further, for any point x^0 in D_0, the sequence $\{x^k\}$, where $x^{k+1} = P(x^k)$, converges to $\bar{x}$ with the following linear rate:*

$$\frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} \leqslant L. \tag{2.2}$$