SENSITIVITY ANALYSIS OF MULTIPLE EIGENVALUES (II)*1)

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Abstract

This paper discusses the sensitivity of semisimple multiple eigenvalues and corresponding invariant subspaces of a complex (or real) $n\times n$ matrix analytically dependent on several parameters. Some results of this paper may be useful for investigating robust multiple eigenvalue assignment in control system design.

§ 1. Introduction

This paper, as a continuation of [7], discusses the sensitivity of semisimple multiple eigenvalues and corresponding invariant subspaces of a complex (or real) $n\times n$ matrix analytically dependent on several parameters. An eigenvalue of a matrix is called semisimple if the maximal degree of the elementary divisors of this eigenvalue is one.

In addition to the notation explained in [7] we use $\mathbb{C}^{m\times n}$ for the set of complex $m\times n$ matrices, $\mathbb{C}^n=\mathbb{C}^{n\times 1}$, $\mathbb{C}=\mathbb{C}^1$ and

$$\mathbb{C}_r^{m\times n}=\{A\in\mathbb{C}^{m\times n}: \operatorname{rank}(A)=r\}.$$

Let $p=(p_1, p_2, \cdots, p_N)^T\in C^N$. Suppose that $A(p)\in\mathbb{C}^{n\times n}$ is an analytic function in some neighbourhood $\mathscr{B}(p^*)$ of the point $p^*\in\mathbb{C}^N$. Without loss of generality we may assume that the point p^* is the origin of $\mathbb{C}^N$. We consider in this paper the eigenproblem

$$A(p)x(p)=\lambda(p)x(p),\quad \lambda(p)\in\mathbb{C},\ x(p)\in\mathbb{C}^n,\ p\in\mathscr{B}(0). \tag{1.1}$$

First of all we investigate an example.

Example 1.1.

$$A(p)=\begin{bmatrix}1+2p_1+2p_2 & p_2\\ 2p_1 & 1+4p_2\end{bmatrix},\quad p=(p_1, p_2)^T\in\mathbb{C}^2. \tag{1.2}$$

Obviously, the matrix $A(p)$ is an analytic function of $p\in\mathbb{C}^2$, $A(0)$ has a multiple eigenvalue 1 and the eigenvalues of $A(p)$ are

$$\lambda_1(p)=1+p_1+3p_2+\sqrt{p_1^2+p_2^2},\quad \lambda_2(p)=1+p_1+3p_2-\sqrt{p_1^2+p_2^2}. \tag{1.3}$$

Observe that by the theory of analytic function of one complex variable the function $\sqrt{z}$ for $z\in\mathbb{C}$ is defined as

$$\sqrt{z}=|z|^{1/2}e^{(i/2)\arg z},\quad \arg z\in(-\pi,\pi];$$

consequently, if we set

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$$\Delta_1=\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right],\quad \Delta_2=\left(-\pi,\ -\frac{\pi}{2}\right]\cup\left(\frac{\pi}{2},\ \pi\right],\tag{1.4}$$

then

$$\frac{\sqrt{z^2}}{z}=\frac{e^{\frac{i}{2}\arg(z^2)}}{e^{i\arg z}}=\begin{cases}1, & \arg z\in\Delta_1,\\ -1, & \arg z\in\Delta_2.\end{cases}\tag{1.5}$$

Utilizing (1.4) and (1.5) we get

$$\left(\frac{\partial\lambda_1(p)}{\partial p_1}\right)_{p=0,\ \arg p_1\in\Delta_1}=\left(\frac{\partial\lambda_2(p)}{\partial p_1}\right)_{p=0,\ \arg p_1\in\Delta_2}=2,\tag{1.6}$$

$$\left(\frac{\partial\lambda_1(p)}{\partial p_1}\right)_{p=0,\arg p_1\in\Delta_2}=\left(\frac{\partial\lambda_2(p)}{\partial p_1}\right)_{p=0,\arg p_1\in\Delta_1}=0,\tag{1.7}$$

$$\left(\frac{\partial\lambda_1(p)}{\partial p_2}\right)_{p=0,\ \arg p_2\in\Delta_1}=\left(\frac{\partial\lambda_2(p)}{\partial p_2}\right)_{p=0,\ \arg p_2\in\Delta_2}=4\tag{1.8}$$

and

$$\left(\frac{\partial\lambda_1(p)}{\partial p_2}\right)_{p=0,\ \arg p_2\in\Delta_2}=\left(\frac{\partial\lambda_2(p)}{\partial p_2}\right)_{p=0,\ \arg p_2\in\Delta_1}=2.\tag{1.9}$$

Here we define

$$\left(\frac{\partial\lambda_s(p)}{\partial p_1}\right)_{p=0,\ \arg p_1\in\Delta_t}=\lim_{\substack{|p_1|\to0\\ \arg p_1=\text{const.}\in\Delta_t}}\frac{\lambda_s(p_1,\ 0)-\lambda_s(0,0)}{p_1},\quad s,\ t=1,\ 2;$$

the partial derivatives $\left(\frac{\partial\lambda_s(p)}{\partial p_2}\right)_{p=0,\ \arg p_2\in\Delta_t}$ $(s,\ t=1,\ 2)$ are defined similarly.

The relations (1.6)—(1.9) show that the functions $\lambda_1(p)$ and $\lambda_2(p)$ are not derivable at $p=0$. Besides, it is worth-while to point out that the functions $\lambda_1(p)$ and $\lambda_2(p)$ are continuous at $p=0$ but not in any neighbourhood of the branch point $p=0$.

Now we set

$$\hat{A}(p_1)=(A(p))_{p=(p_1,0)^T},\quad \tilde{A}(p_2)=(A(p))_{p=(0,p_2)^T},$$

in which $A(p)$ is described in (1.2). It is easy to see that $\hat{A}(0)$ and $\tilde{A}(0)$ have multiple eigenvalue 1, the eigenvalues of $\hat{A}(p_1)$ are

$$\hat{\lambda}_1(p_1)=1+2p_1,\quad \hat{\lambda}_2(p_1)=1,\tag{1.10}$$

and the eigenvalues of $\tilde{A}(p_2)$ are

$$\tilde{\lambda}_1(p_2)=1+2p_2,\quad \tilde{\lambda}_2(p_2)=1+4p_2.\tag{1.11}$$

Comparing (1.10), (1.11) with (1.3) we find that

$$(\lambda_1(p))_{p=(p_1,0)^T}=\begin{cases}\hat{\lambda}_1(p_1), & \arg p_1\in\Delta_1,\\ \hat{\lambda}_2(p_1), & \arg p_1\in\Delta_2,\end{cases}$$

$$(\lambda_2(p))_{p=(p_1,0)^T}=\begin{cases}\hat{\lambda}_2(p_1), & \arg p_1\in\Delta_1,\\ \hat{\lambda}_1(p_1), & \arg p_1\in\Delta_2,\end{cases}$$

$$(\lambda_1(p))_{p=(0,p_2)^T}=\begin{cases}\tilde{\lambda}_2(p_2), & \arg p_2\in\Delta_1,\\ \tilde{\lambda}_1(p_2), & \arg p_2\in\Delta_2,\end{cases}$$

$$(\lambda_2(p))_{p=(0,p_2)^T}=\begin{cases}\tilde{\lambda}_1(p_2), & \arg p_2\in\Delta_1,\\ \tilde{\lambda}_2(p_2), & \arg p_2\in\Delta_2,\end{cases}$$

where Δ_1 and Δ_2 are defined by (1.4).

We note that the following facts are important: the functions $\hat{\lambda}_1(p_1)$ and $\hat{\lambda}_2(p_1)$