A CHARACTERISTIC ITERATION FOR SOLVING COEFFICIENT INVERSE PROBLEMS OF A WAVE EQUATION*

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Abstract

In this paper, a characteristic iteration for solving coefficient inverse problems has been presented. This method is stable and fast converging, and may be extended to the 2-D case. Excellent numerical results have been obtained by this method.

§ 1. 1-D Wave Equation and Its Inverse Problem

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial u}{\partial x}\right)-\sigma(x)\frac{\partial^2 u}{\partial t^2}=0, \quad x>0,\ t>0, \tag{1.1}$$

$$u(x,0)=\frac{\partial u}{\partial t}(x,0)=0, \quad x>0, \tag{1.2}$$

$$\frac{\partial u}{\partial x}(0,t)=\delta(t), \quad t>0. \tag{1.3}$$

Measured data:

$$u(0,t)=f(t). \tag{1.4}$$

Inverse problem:

To recover $\sigma(x)$ from $f(t)$ and (1.1)—(1.3), where $\sigma(x)$ is the coefficient of (1.1), which belongs to

$$\Sigma=\{\sigma(x)\,|\,\sigma(x)\in C^1(0,\infty),\ 0<\underline{\sigma}\leqslant\sigma(x)\leqslant\bar{\sigma}\}$$

and $\delta(t)$ is the generalized Delta function.

Assume that $f(t)$ satisfies a certain compatible condition, for example, $f(0)=-1$, such that the solution of the inverse problem exists.

§ 2. Singularity of the Solution of (1.1)—(1.3)

Lemma 1. *Suppose that $\sigma(x)$ belongs to Σ and $u(x,t)$ is the generalized solution of* (1.1)—(1.3). *Then,*

$$u(x,t)=a(x)H(t-x)+v(x,t), \tag{2.1}$$

where $a(x)$ is the jump quantity, $H(\cdot)$ is a Heaviside function, and $v(x,t)$ is the

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regular part of $u(x, t)$. Moreover, $a(x)$ and $v(x, t)$ satisfy respectively

$$a(x) = -\frac{\sqrt{\sigma(0)}}{\sqrt{\sigma(x)}}, \tag{2.2}$$

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right) - \sigma(x)\frac{\partial^2 v}{\partial t^2} = -\left[\sigma(x)\frac{d^2 a}{dx^2} + \frac{d\sigma}{dx}\frac{da}{dx}\right]H(t-x), \tag{2.3}$$

$$v(x, 0) = v_t(x, 0) = 0, \tag{2.4}$$

$$v_x(0, t) = -\frac{\sigma'(0)}{\sigma(0)}H(t). \tag{2.5}$$

Proof. Since $\sigma(x) \in \Sigma$, (1.1) is a wave equation. By [3], the singularities of the solution of (1.1)—(1.3) propagate along the characteristic line $t=x$. By (1.2) and (1.3), we have (2.1).

Since $u(x, t)$ is the generalized solution of (1.1)—(1.3), it should satisfy the weak equation of (1.1)—(1.3). By the theory of generalized function[3], we can make generalized calculus on both sides of (2.1). Thus we have

$$\frac{\partial u}{\partial x} = -a(x)\delta(t-x) + a_x H(t-x) + \frac{\partial v}{\partial x}, \tag{2.6}$$

$$\frac{\partial^2 u}{\partial x^2} = a(x)\delta'(t-x) - 2a_x\delta(t-x) + a_{xx}H(t-x) + \frac{\partial^2 v}{\partial x^2}, \tag{2.7}$$

$$\frac{\partial u}{\partial t} = a(x)\delta(t-x) + \frac{\partial v}{\partial t}, \tag{2.8}$$

$$\frac{\partial^2 u}{\partial t^2} = a(x)\delta'(t-x) + \frac{\partial^2 v}{\partial t^2}. \tag{2.9}$$

Substituting (2.6)—(2.9) into (1.1) and making proper arrangement, we have

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right) - \sigma(x)\frac{\partial^2 v}{\partial t^2} = (2a_x\sigma(x) + a(x)\sigma_x)\delta(t-x) - (\sigma(x)a_{xx} + \sigma_x a_x)H(t-x). \tag{2.10}$$

Let

$$R(x, t) = \frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right) - \sigma(x)\frac{\partial^2 v}{\partial t^2}, \tag{2.11}$$

where $R(x, t)$ does not include the δ function. This is because by (2.1) we have

$$v(x, t) = b(x)C(t-x) + w(x, t), \tag{2.12}$$

in which $w(x, t)$ belongs to C^1 and $C(t-x) = (t-x)^+$. Clearly, we have

$$R(x, t) = (-2b_x\sigma(x) - b(x)\sigma_x)H(t-x) + (\sigma(x)b_{xx} + \sigma_x b_x)C(t-x) + \frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial w}{\partial x}\right) - \sigma(x)\frac{\partial^2 w}{\partial t^2}. \tag{2.13}$$

In (2.13), since $w(x, t)$ belongs to C^1, it is obvious that there is no δ function in $\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial w}{\partial x}\right) - \sigma(x)\frac{\partial^2 w}{\partial t^2}$. Thus the coefficient of $\delta(t-x)$ in (2.10) should vanish, i.e.

$$2a_x\sigma(x) + a(x)\sigma_x = 0, \tag{2.14}$$

and we have

$$\frac{\partial}{\partial x}\left(\sigma(x)\frac{\partial v}{\partial x}\right) - \sigma(x)\frac{\partial^2 v}{\partial t^2} = -(\sigma(x)a_{xx} + \sigma_x a_x)H(t-x),$$