THE THEORY OF FILLED FUNCTION METHOD FOR FINDING GLOBAL MINIMIZERS OF NONLINEARLY CONSTRAINED MINIMIZATION PROBLEMS*

GE REN-PU (葛人溥)
(*Institute of Computational and Applied Mathematics, Xi'an Jiaotong University, Xi'an, China*)

Abstract

This paper is an extension of [1]. In this paper the descent and ascent segments are introduced to replace respectively the descent and ascent directions in [1] and are used to extend the concepts of S-basin and basin of a minimizer of a function. Lemmas and theorems similar to those in [1] are proved for the filled function

$$P(x, r, \rho)=\frac{1}{r+F(x)}\exp(-\|x-x_1^*\|^2/\rho^2), \tag{0.1}$$

which is the same as that in [1], where x_1^* is a constrained local minimizer of the problem (0.3) below and

$$F(x)=f(x)+\sum_{i=1}^{m'}\mu_i|c_i(x)|+\sum_{i=m'+1}^{m}\mu_i\max(0, -c_i(x)) \tag{0.2}$$

is the exact penalty function for the constrained minimization problem

$$\min_x f(x),$$

subject to

$$\begin{aligned} &c_i(x)=0, \quad i=1, 2, \cdots, m', \\ &c_i(x)\geqslant 0, \quad i=m'+1, \cdots, m, \end{aligned} \tag{0.3}$$

where $\mu_i>0$ ($i=1, 2, \cdots, m$) are sufficiently large. When x_1^* has been located, a saddle point or a minimizer $\hat{x}$ of $P(x, r, \rho)$ can be located by using the nonsmooth minimization method with some special termination principles. The $\hat{x}$ is proved to be in a basin of a lower minimizer x_2^* of $F(x)$, provided that the ratio $\rho^2/[r+F(x_1^*)]$ is appropriately small. Thus, starting with $\hat{x}$ to minimize $F(x)$, one can locate x_2^*. In this way a constrained global minimizer of (0.3) can finally be found and termination will happen.

§ 1. Introduction

Many existing methods for constrained minimization problems are only used for finding a constrained local minimizer. This paper develops a method which can be used to find the constrained global minimizer. This method is based on two existing methods. One is the nonsmooth exact penalty function method, which transforms a constrained minimization problem

$$\min_x f(x),$$

subject to

$$\begin{aligned} &c_i(x)=0, \quad i=1, 2, \cdots, m', \\ &c_i(x)\geqslant 0, \quad i=m'+1, \cdots, m \end{aligned} \tag{1.1}$$

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into a nonsmooth unconstrained minimization problem[2]

$$\min_x F(x)=f(x)+\sum_{i=1}^{m'}\mu_i|c_i(x)|+\sum_{i=m'+1}^{m}\mu_i\max[0,\ -c_i(x)], \tag{1.2}$$

where $\mu_i>0$ ($i=1, 2, \cdots, m$) are sufficiently large and $F(x)$ is not differentiable on the surfaces

$$c_i(x)=0,\quad i=1, 2, \cdots, m. \tag{1.3}$$

The other is the filled function method for finding global minimizers of a continuously differentiable function[1], which can be extended to the nonsmooth minimization case. Here the nonsmoothness means that the objective function has no continuous gradient.

Thus, in Section 2 an extension of the filled function method to the nonsmooth case is developed. Section 3 gives an algorithm which uses the extension above to solve the constrained global minimization problems. Section 4 is a brief discussion.

This paper is closely related to [1]. Therefore it is not necessary to repeat the same context and only the different points will be mentioned in this paper.

By the way, it is still assumed that the functions $f(x)$ and $c_i(x)$ in (1.2) and (1.3) are all twice continuously differentiable.

§ 2. The Global Minimization of a Nonsmooth Function

This section is concerned with the problem of finding a global minimizer of a nonsmooth function. The definitions 1 and 2 in [1] should be changed in the following way.

Definition 1. *A segment x_1-x_2 is said to be a descent segment of $F(x)$ if the inequality*

$$F(x_1)<F(\alpha x_2+(1-\alpha)x_1)<F(\beta x_2+(1-\beta)x_1)<F(x_2) \tag{2.1}$$

holds for $x_1\neq x_2$ and any α, β satisfying

$$0<\alpha<\beta<1. \tag{2.2}$$

If the inequality opposite to (2.1) holds, then x_1-x_2 is said to be an ascent segment of $F(x)$.

Definition 2. *Suppose x_1^* is a minimizer of $F(x)$, where $F(x)$ is a nonsmooth function. The S-basin of $F(x)$ at x_1^* is a connected set S_1^*, which contains x_1^* and in which for any point x the segment x_1^*-x is a descent segment of $F(x)$, provided $x\neq x_1^*$.*

Definition 3. *Suppose x_1^* is a minimizer of $F(x)$. The basin of $F(x)$ at x_1^* is a connected set B_1^* which contains the S-basin S_1^* at x_1^* and for any $x\in B_1^*$ there exist a finite number of points $x_i\in B_1^*$ ($i=1, 2, \cdots, m$) (it is allowed that $x=x_i$) but $x_m\in S_1^*$, such that the inequalities*

$$F(x_i)<F(\alpha_i x_{i-1}+(1-\alpha_i)x_i)<F(\beta_i x_{i-1}+(1-\beta_i)x_i)<F(x_{i-1}) \tag{2.3}$$

hold for $i=1, 2, \cdots, m$, $x_0=x$, $0<\alpha_i<\beta_i<1$. If x_1^* is a maximizer of $G(x)$, then the hill of $G(x)$ at x_1^* is the basin of the function $-G(x)$.

These definitions clearly imply the following lemmas.

Lemma 1. *There exists a descent route which leads any $x\in B_1^*$ to descend to x_1^*.*

Lemma 2. *Suppose B_1^* is the basin of $F(x)$ at x_1^*. Then the inequality*