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ON THREE-POINT SECOND-ORDER ACCURATE CONSERVATIVE DIFFERENCE SCHEMES*1)

TANG TAO (汤 涛)

(*Department of Mathematics, Peking University, Beijing, China*)

§1. Introduction

In this paper we study 3-point 2nd-order accurate conservative TSC difference schemes. The TSC schemes are named after their three computational features:

(A) They are 3-point schemes. Only three points are needed to determine a point on the next time level. So they can fit conveniently the boundary conditions for initial boundary value problems.

(B) They have 2nd-order accuracy for smooth solutions. So in the smooth parts of the solution, we can get a better numerical result.

(C) They are in conservation form. By the Lax-Wendroff theorem[10], if the computed solution of the TSC schemes converges boundedly a.e. to u, then u is a weak solution of Eq. (2.1).

In this paper we prove that the TSC scheme is not TVNI (total variation nonincreasing). We prove that the 3-point s-order accuracy ($s\geqslant 2$) linear difference scheme is linearly $l_p(1\leqslant p\leqslant +\infty,\ p\neq 2)$ unstable. So a linear TSC scheme is linearly $l_p(1\leqslant p\leqslant +\infty,\ p\neq 2)$ unstable. In addition, a rigorous proof of the nonlinear l_2 instabilities for the two-step Richtmyer[6] scheme is given. At last, a successful modification to the Lax-Wendroff scheme, the Richtmyer scheme and the MacCormack scheme is got. The modified schemes retain their computational features (A), (B) and (C) mentioned above, with an additional property that the limit solutions satisfy the entropy condition for all the convex smooth flux functions f. They are l_2 stable in the sense of Definition 3.2 when we choose $\theta(s)\equiv 1$ in the modified schemes (3.7), (3.9) and (3.10).

§2. The stability results for TSC schemes

The simplest mathematical models of inviscid compressible fluid dynamics are given by solutions, u, of the scalar convex conservation law

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial f(u)}{\partial x}\equiv\dfrac{\partial u}{\partial t}+a(u)\dfrac{\partial u}{\partial x}=0, \\ u(x,\,0)=u_0(x), \end{cases} \tag{2.1}$$

where f is a smooth convex function of u.

We shall discuss numerical approximations to weak solutions of (2.1) which

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are obtained by $(2k+1)$-point explicit schemes in conservation form,

$$u_j^{n+1}=u_j^n-\lambda(g_{j+\frac{1}{2}}^n-g_{j-\frac{1}{2}}^n), \tag{2.2a}$$

where

$$g_{j+\frac{1}{2}}^n=g(u_{j-k+1}^n, \cdots, u_{j+k}^n). \tag{2.2b}$$

Here $u_j^n=\bar{u}(j\Delta x, n\Delta t)$ and g is a continuous numerical flux function. We require that the numerical flux function be consistent with the flux function $f(u)$ in the following sense

$$g(u, \cdots, u)=f(u). \tag{2.2c}$$

The class of schemes we shall discuss can be written in the form

$$u_j^{n+1}=u_j^n+d_{+,j+\frac{1}{2}}^{(2k+1)}\Delta_+u_j^n-d_{-,j-\frac{1}{2}}^{(2k+1)}\Delta_+u_{j-1}^n, \tag{2.3a}$$

where

$$d_{+,j+\frac{1}{2}}^{(2k+1)}=\begin{cases}\lambda\dfrac{f_j-g_{j+\frac{1}{2}}}{\Delta_+u_j}, & \text{if } \Delta_+u_j\neq 0,\\ -\lambda a(u_j), & \text{if } \Delta_+u_j=0\end{cases} \tag{2.3b}$$

and

$$d_{-,j-\frac{1}{2}}^{(2k+1)}=\begin{cases}\lambda\dfrac{f_j-g_{j-\frac{1}{2}}}{\Delta_+u_{j-1}}, & \text{if } \Delta_+u_{j-1}\neq 0,\\ \lambda\cdot a(u_j), & \text{if } \Delta_+u_{j-1}=0.\end{cases} \tag{2.3c}$$

Here and throughout this paper, we use the standard notation

$$\Delta_+u_j=u_{j+1}-u_j;\ f_j=f(u_j);\ \lambda=\Delta t/\Delta x.$$

Scheme (2.3) includes the TSC scheme and Harten's TVNI scheme[5].

We say that the finite difference scheme (2.2) is total variation nonincreasing TVNI if for all nonnegative integers n, we have

$$TV(u^{n+1})\leqslant TV(u^n), \tag{2.4a}$$

where

$$TV(u^n)=\sum_{j=-\infty}^{+\infty}|\Delta_+u_j^n|.$$

Theorem 2.1. *If the scheme* (2.1) *is* TVNI, *then we have the following inequalities,*

$$\frac{f(a)-g(b, a, \cdots, a)}{b-a}\leqslant 0, \tag{2.5a}$$

$$\frac{f(a)-g(a, \cdots, a, b)}{b-a}\geqslant 0 \tag{2.5b}$$

for all real numbers a and b, $a\neq b$.

Proof. If $b>a$, we assume

$$u_j^n=\begin{cases}a, & j<J,\\ b, & j\geqslant J,\end{cases} \tag{2.6}$$

where J is an arbitrarily fixed integer. If $u_{J-k}^{n+1}<a$, then

$$TV(u^{n+1})\geqslant|u_{J-k}^{n+1}-u_{J+2k+2}^{n+1}|=|u_{J-k}^{n+1}-b|>b-a=TV(u^n),$$