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ON THE CARDINALITIES OF INTERPOLATING RESTRICTED RANGES*

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Abstract

The cardinalities of interpolating restricted range $\boldsymbol{R}_1(m, n)$ for best rational approximation are discussed. The conditions under which the cardinality of $\boldsymbol{R}_1(m, n)$ is 0 or 1 or ∞ are established by using the results obtained in [3, 4].

The problem of best rational approximation with interpolating constraints proposed by Wang (cf. [1], p. 69) can be transformed into an equivalent form by means of Leibniz Rule for derivatives. The problem can be stated as follows.

Assume that $f\in C[a, b]$, $w\in C^t[a, b]$ with $w>0$, and $\boldsymbol{R}(m, n)$ is the class of rational functions

$$\boldsymbol{R}(m, n)=\{R\colon R=P/Q,\ P\in H_m,\ Q\in H_n\backslash\{0\}\}.$$

Let a set of $s+1$ distinct points in $[a, b]$

$$a\leqslant x_0<x_1<\cdots<x_s\leqslant b$$

and a corresponding set of real numbers

$$y_i^{(j)}\quad (i=0, 1, \cdots, s;\ j=0, 1, \cdots, k_i)$$

be given, where

$$0\leqslant k_i\leqslant t,\quad i=0, 1, \cdots, s,\quad k\equiv\sum_{i=0}^{s}(k_i+1)\leqslant m+n.$$

Find a rational function $R\in\boldsymbol{R}_1(m, n)$ such that

$$\|f-wR\|_{[a,b]}=\inf\{\|f-wr\|_{[a,b]}\colon r\in\boldsymbol{R}_1(m, n)\},\tag{1}$$

where $\boldsymbol{R}_1(m, n)\subset\boldsymbol{R}(m, n)$ is the set consisting of all irreducible rational functions satisfying the following constraints

$$\boldsymbol{R}^{(j)}(x_i)\equiv\frac{d^jR(x)}{dx^j}\bigg|_{x=x_i}=y_i^{(j)},\quad i=0, 1, \cdots, s;\ j=0, 1,\cdots, k_i.\tag{2}$$

$\boldsymbol{R}_1(m, n)$ is called the interpolating restricted range.

Solvability of this problem has already been investigated. Especially, the existence, uniqueness and characteristics of its solution have been discussed. It would be worthwhile to call attention to the fact that all the related discussions follow as results from a tacit assumption, namely, $\boldsymbol{R}_1(m, n)$ remains nonempty. However, it turns out that this is not always the case. A simple example can illustrate it well. Taking $k_i=0$ $(i=0, \cdots, s)$, $k>m+1$ and

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$$y_i^{(0)}=\begin{cases}0, & i=0, 1, \cdots, m,\\ 1, & i=m+1, \cdots, k-1,\end{cases}$$

it is obvious that, for any aforesaid set of the x_i's $\in[a, b]$, there is no rational function $R\in \boldsymbol{R}(m, n)$ satisfying

$$R(x_i)=y_i^{(0)}, \quad i=0, 1, \cdots, k-1,$$

i.e., $\boldsymbol{R}_1(m, n)=\emptyset$, the empty set.

The aim of our paper is simply to propound as a top priority regarding best rational approximation the fundamental question of under what conditions that $\boldsymbol{R}_1(m, n)$ will or will not be empty. We note that such a proposal is equivalent to considering the solvability of an under determined rational interpolation problem, which is clearly expressed by (2). Here "under determined" means that the number of interpolating conditions is less than that of the parameters to be determined. In what follows all our discussions and conclusions are based upon [4]. For the sake of brevity and convenience, we shall follow closely the notations used in [4] without any specification.

We first state a lemma concerning the interpolating properties of quasi-rational interpolant r^*_{mn}. This lemma includes both Theorem 3.2 and Theorem 3.5 given in [4] as its special cases.

Lemma 1. *Assume that* $R\in \boldsymbol{R}_0(m, n)$, $0\leqslant l_i\leqslant k_i+1$ $(i=0, 1, \cdots, s)$, $\mu\leqslant p\leqslant m$, $\nu\leqslant q\leqslant n$. *Then* $x_i\in X$ *is an* l_i*-fold unattainable point of* R *if and only if*

(i) rank $C_i^{(j)}$ $(p-1, q-1)<$rank $C(p-1, q-1)$, $j=k_i, k_i-1, \cdots, k_i-l_i+1$;

(ii) rank $C_i^{(k_i-l_i)}(p-1, q-1)=$rank $C(p-1, q-1)$.

The proof of this lemma is similar to that of Theorem 3.5 in [4]; the only thing we have to do is to replace m and n therein with p and q respectively. We note that the case $(p, q)=(\mu, \nu)$ corresponds to Theorem 3.2, and likewise, $(p, q)=(m, n)$ to Theorem 3.5.

Let us now consider the cardinality of $\boldsymbol{R}_1(m, n)$. To start with, we introduce the following

Lemma 2. *Let* P_1/Q_1, $P_2/Q_2\in \boldsymbol{R}_1(m, n)$ *with* $P_1/Q_1\neq P_2/Q_2$. *For any pair* (α, β) *of constants, if*

$$\alpha Q_1(x_i)+\beta Q_2(x_i)\neq 0, \quad i=0, 1, \cdots, s,$$

then

$$\frac{\alpha P_1+\beta P_2}{\alpha Q_1+\beta Q_2}\in \boldsymbol{R}_1(m, n).$$

Proof. Let $g\in C^t[a, b]$ satisfy

$$g^{(j)}(x_i)=y_i^{(j)}, \quad i=0, 1, \cdots, s; \quad j=0, 1, \cdots, k_i.$$

Then, from

$$(P_r-gQ_r)^{(j)}(x_i)=0, \quad i=0, 1, \cdots, s; j=0, 1, \cdots, k_i; r=1, 2,$$

we have

$$[(\alpha P_1+\beta P_2)-g(\alpha Q_1+\beta Q_2)]^{(j)}(x_i)=0, \quad i=0, 1, \cdots, s; j=0, 1, \cdots, k_i.$$

Since $\alpha Q_1(x_i)+\beta Q_2(x_i)\neq 0$ $(i=0, 1, \cdots, s)$, it follows that

$$\left(\frac{\alpha P_1+\beta P_2}{\alpha Q_1+\beta Q_2}\right)^{(j)}(x_i)=g^{(j)}(x_i)=y_i^{(j)}, \quad i=0, 1, \cdots, s; j=0, 1, \cdots, k_i.$$