NONCONFORMING FINITE ELEMENTS WITH COMPENSATION FOR PLATE BENDING PROBLEMS*

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§ 1. Introduction

As many authors pointed out, to solve plate bending problems and other elliptic boundary value problems of equations with order higher than two it is much more convenient to use nonconforming finite elements, for they have less degrees of freedom, relatively simpler base functions and better convergence behavior when convergence occurs. There are two methods, among others, which use the nonconforming elements. One is the so called tolerance method[3], in which we calculate the bilinear form of the variational problem element by element and then take the sum (briefly, elements sum) as the value of the bilinear function on the whole domain, just as what we do in using conforming elements. With this method, the convergence of the finite element approximations should be analysed carefully, since they are not always convergent to the solution of the original problem as the mesh gets finer. For the method many elements have been analysed, see, for example, [2, 5, 7] and [9]. But all these analyses show that the convergence order is lower than that of the conforming elements with the same degree of piecewise polynomial interpolation. This is because the elements sum is used to substitute the real bilinear form. The other method is the penalty method. As shown in [1, 3] and [8], the convergence always occurs, but its order is only half of that of the conforming elements with the same degree of piecewise polynomials.

To improve the accuracy order, we will give a compensation method in this paper. The main idea of the method is to add something to the elements sum so that the error caused by the substitution of this sum for the bilinear function can be compensated. This method gives better approximations than both the tolerance method and the penalty method under certain conditions. Moreover, if the elements used are the so called weakly discontinuous elements[3], the method gives approximations of the same accuracy order with that of the conforming elements. In addition it has the advantage that no Lagrange multipliers or other additional parameters are used, so no additional degrees of freedom will be introduced. Hence the amount of computations will not be increased.

The paper is outlined as follows: In § 2, a variational model with compensation of the clamped plate bending problem will be described, and the existence and uniqueness of the solution of the proposed variational problem are analysed. In § 3, the error estimates are given and the theorem of convergence is proved. In § 4, we

* Received November 5, 1985.

will deal with plates with other kinds of boundary conditions. Finally, some examples are given in § 5.

§ 2. The Compensation Method for Clamped Plates

Let us consider an elastic thin plate of a convex polygonal domain Ω in R^2. A mathematical model of the bending problem of the plate with clamped boundary is

$$\begin{cases} \Delta^2 u = f, & (x_1, x_2) \in \Omega, & (2.1) \\ u = \dfrac{\partial u}{\partial n} = 0, & (x_1, x_2) \in \partial\Omega, & (2.2) \end{cases}$$

where n is the outward normal direction of boundary $\partial\Omega$. Set

$$a(u, v) = \int_\Omega (u_{x_1x_1} v_{x_1x_1} + 2u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2})\, dx,$$

where $dx = dx_1\, dx_2$, $u_{x_1x_2} = \dfrac{\partial^2 u}{\partial x_1 \partial x_2}$, etc. We can associate with (2.1)—(2.2) a variational problem:

$$\text{Find } u \in H_0^2(\Omega), \text{ such that } a(u, v) = (f, v)_0 = \int_\Omega f v\, dx, \quad \forall v \in H_0^2(\Omega). \tag{2.3}$$

As is well known, $a(u, v)$ is $H_0^2(\Omega)$-elliptic, i.e. there exists a constant $m > 0$ independent of u, such that

$$m\|u\|_{2,\Omega}^2 \leqslant a(u, u), \quad \forall u \in H_0^2(\Omega). \tag{2.4}$$

Also, $a(u, v)$ is bounded in $H^2(\Omega)$, i.e. there exists a constant M independent of u and v, such that

$$|a(u, v)| \leqslant M \|u\|_{2,\Omega} \|v\|_{2,\Omega}.$$

Hence problem (2.3) has a unique solution u, which is called the weak solution of (2.1)—(2.2).

We are going to consider solving (2.3) approximately by the finite element method. As usual, put a triangulation on Ω and let Ω_h be the set of all triangles (elements) obtained. For $\sigma \in \Omega_h$, set

h_σ = the diameter of σ,

ρ_σ = the diameter of the inscribed circle in σ.

Let

$$h = \max_{\sigma \in \Omega_h} h_\sigma, \quad \rho = \min_{\sigma \in \Omega_h} \rho_\sigma.$$

Next, we will consider a family of triangulations on Ω, which will be called a regular family if each of its triangulation satisfies

$$\frac{h}{\rho} \leqslant C, \tag{2.5}$$

a constant being independent of the triangulation. By the way, letters c and C will be used as generic constants which may take different values at different places. We will always assume that $h \leqslant 1$. Now, construct a finite element space (i.e. a piecewise polynomial space) V_h associated with the decomposition of Ω. Let $V_h(\sigma)$ be the set of all restrictions on σ of functions in V_h and $P_k(\sigma)$ the set of all polynomials of degree $\leqslant k$ on σ. Suppose $P_k(\sigma) \subset V_h(\sigma)$. In general, we do not make the assumption $V_h \subset H_0^2(\Omega)$, i.e. V_h is a nonconforming finite element space. For $v \in V_h$, set