A MONOTONICALLY CONVERGENT ITERATIVE METHOD FOR LARGE SPARSE NONLINEAR EQUATIONS*

QIN MENG-ZHAO (秦孟兆)
(Computing Center, Academia Sinica, Beijing, China)

§ 0. Introduction

The bisection method is a well-known method for the numerical solution of a single nonlinear equation. This method is effective and simple in finding the real root of a single nonlinear equation, and only requires that the function be continuous. Therefore it has a wide range of applications. This paper intends to extend this method to the case of nonlinear systems. Although not all nonlinear systems can be solved by the bisection method, there does exist some class of nonlinear systems of equations which can be solved by the bisection method. And this class of systems of equations can be obtained by approximating partial differential equations using the finite element or difference method. For some classes of nonlinear systems the bisection method is simple, safe and reliable[6]. By "safe and reliable" is meant that the desired solution can always be found (in the sence of global convergence).

The paper is built up as follows.

In the first section some definitions and notations are given. Section 2 describes the bisection method and gives its algorithm and its error estimate. Finally this method is applied to the minimal surface problem and some numerical results are given.

§ 1. Definition and Notation

We consider the following nonlinear system of equations

$$Fx=0, \tag{1.1}$$

where

$$Fx=\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

$$x\in D\subset R^n\to R^n.$$

Recall that the natural partial ordering on R^n is defined by $x\leqslant y$ $(x<y)$, for any

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$x, y \in R^n$, if and only if $x_i \leqslant y_i (x_i < y_i)$, $i=1, 2, \cdots, n$.

Definition 1. *The mapping F: $D \subset R^n \to R^m$ is isotone (on D) if for any $x, y \in D$, $x \leqslant y$, implies that $Fx \leqslant Fy$. An isotone mapping F is strictly isotone if it follows from $x < y$, for any $x, y \in D$, that $Fx < Fy$.*

Definition 2. *The mapping F: $D \subset R^n \to R^m$ is antitone (on D) if for any $x, y \in D$, $x \leqslant y$, implies that $Fx \geqslant Fy$. An antitone mapping F is strictly antitone if it follows from $x < y$, for any $x, y \in D$, that $Fx > Fy$.*

Definition 3[2]. *A mapping F: $R^n \to R^n$ is diagonally isotone if, for any $x \in R^n$, the n functions*

$$\psi_{ii}: R^1 \to R^1,\ \psi_{ii}(t) = f_i(x + te^i),\quad i=1, 2, \cdots, n \tag{1.2}$$

are isotone, where e^i are unit vectors. The function F is strictly diagonally isotone if, for any $x \in R^n$, $\psi_{ii} (i=1, 2, \cdots, n)$ are strictly isotone.

Definition 4. *A mapping F: $R^n \to R^n$ is off-diagonally antitone if, for any $x \in R^n$, the functions*

$$\psi_{ij}: R^1 \to R^1,\ \psi_{ij}(t) = f_i(x + te^j),\quad i \neq j,\ i, j=1, 2, \cdots, n \tag{1.3}$$

are antitone.

§ 2. Bisection Method

In Fig. 1, assuming $\varphi_i(x_i^{(k)}) < 0$, $\psi_i(y_i^{(k)}) > 0$, we first define a sequence of intervals:

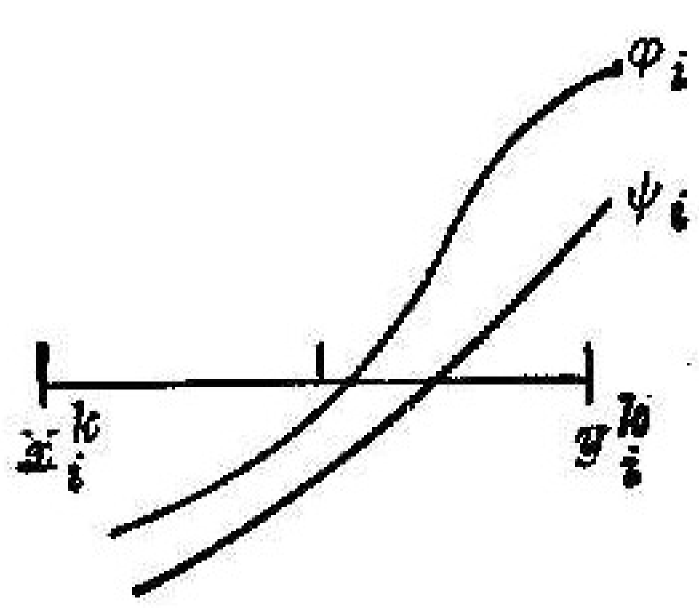

Fig. 1

$$I_{ij}^{(k)} = (x_i^{(k)}, x_{i,j+1}^{(k)}),$$

where

$$x_{ij}^{(k)} = \frac{x_i^{(k)} + x_{i,j-1}^{(k)}}{2},\quad j=1, 2, \cdots, m_i,\ x_{i0}^{(k)} := y_i^{(k)}. \tag{2.1}$$

Continue this process until

$$\varphi_i(x_i^{(k)})\, \varphi_i\left(\frac{x_i^{(k)} + x_{ij}^{(k)}}{2}\right) > 0. \tag{2.2}$$

Let m_i be the first index for $\varphi_i(x_{ij}^{(k)}) < 0$, and set

$$x_{i,m_i}^{(k)} = x_i^{(k+1)},$$

$$\left(x_i^{(k)}, \frac{x_i^{(k)} + y_i^{(k)}}{2}\right) - I_{i,m_i}^{(k)} = I_i^{(k)}. \tag{2.3}$$

Analogously, a sequence of intervals

$$R_{ij}^{(k)} = (y_{i,j+1}^{(k)}, y_i^{(k)}),$$

where

$$y_{ij}^{(k)} = \frac{y_{ij-1}^{(k)} + y_i^{(k)}}{2},\ j=1, 2, \cdots, n_i,\ y_{i0}^{(k)} := x_i^{(k)}, \tag{2.4}$$

can be defined until

$$\psi_i(y_i^{(k)})\, \psi_i\left(\frac{y_{ij}^{(k)} + y_i^{(k)}}{2}\right) > 0. \tag{2.5}$$

Let n_i be the first index for $\psi_i(y_{ij}^{(k)}) > 0$, and set

$$y_{i,n_i}^{(k)} = y_i^{(k+1)},$$