ERROR ESTIMATES OF TWO NONCONFORMING FINITE ELEMENTS FOR THE OBSTACLE PROBLEM*

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Abstract

The linear nonconforming element and Wilson's element for the obstacle problem are considered. Optimal error bounds for both elements are obtained in the case of regular subdivisions of domain Ω in R^2.

§ 1. Abstract Error Estimate

There are a number of works in the analysis of finite element methods for variational inequalities (c.f. [7] and the references therein). Particularly the analysis of F.E.M for the obstacle problem has been studied more or less completely (c.f. [2]—[9]). All these analyses, however, are related to conforming finite elements, except for the mixed type[3].

In this paper, we will analyse two nonconforming finite elements for the obstacle problem. We first show an abstract error estimate, which is similar to the Second Strang Lemma[5]. Next, in § 2, we will analyse the linear nonconforming element approximation to the obstacle problem. Finally, in § 3, Wilson's element will be considered for the obstacle problem.

Let Ω be a convex domain in R^2 with piecewise smooth boundary $\partial\Omega$, X a Hilbert space of functions defined on Ω with norm $\|\cdot\|$, K a nonempty convex closed subset in X, and $a(\cdot, \cdot)$ a continuous, X-elliptic, bilinear from on $X\times X$, $f\in X'$——the dual space of X, with the duality pairing $\langle\cdot, \cdot\rangle$ between X' and X. The abstract variational inequality considered is the following:

$$\begin{cases} \text{find } u\in K, \text{ such that} \\ a(u, v-u)\geqslant\langle f, v-u\rangle \quad \forall v\in K. \end{cases} \tag{1.1}$$

The solution of (1.1) will be approximated by the finite element method for a regular subdivision. For each $h>0$, let $\mathscr{T}_h$ be a regular subdivision on Ω[5], $\Omega^h=\bigcup_{\tau\in\mathscr{T}_h}\tau$, X_h be a finite element approximate space of X with norm $\|\cdot\|_h$ (either conforming or nonconforming, i.e., $X_h\subset X$ or $X_h\not\subset X$ respectively), and K_h be a convex closed subset in X_h, as an approximation of K. Then the approximate problem of (1.1) is the following:

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$$\begin{cases} \text{find } u_h \in K_h, \text{ such that} \\ a_h(u_h, v_h - u_h) \geqslant \langle f, v_h - u_h \rangle_h \quad \forall v_h \in K_h, \end{cases} \tag{1.2}$$

where

$$a_h(u_h, v_h) = \sum_{\tau \in \mathscr{T}_h} a(u_h|_\tau, v_h|_\tau),$$
$$\langle f, v_h \rangle_h = \sum_{\tau \in \mathscr{T}_h} \langle f|_\tau, v_h|_\tau \rangle,$$

and $u_h|_\tau$, $v_h|_\tau$ and $f|_\tau$ are the restrictions of u, v and f on the element τ respectively.

Throughout this paper we will use the notations of Sobolev spaces $H^m(\Omega)$ as in [1] and we assume that C is a generic constant, which may have different values in different places, if not specifically indicated.

We have an abstract error estimate, similar to Second Strang Lemma[5], as follows:

Theorem 1. *Assume that $a_h(\cdot, \cdot)$ is a continuous, X_h-elliptic, bilinear form on $X_h \times X_h$, and u and u_h are the solutions of problems (1.1) and (1.2) respectively. Then there exists a constant C independent of X_h such that*

$$\|u - u_h\|_h < C \inf_{v_h \in K_h} \left\{ \|u - v_h\|_h + \frac{a_h(u, v_h - u_h) - \langle f, v_h - u_h \rangle_h}{\|u_h - v_h\|_h} \right\}. \tag{1.3}$$

The proof is easy and similar to that in [5], so it is omitted.

§ 2. The Linear Nonconforming Element

To begin with, we consider the obstacle problem:

$$\begin{cases} \text{find } u \in K, \text{ such that} \\ a(u, v-u) \geqslant \langle f, v-u \rangle \quad \forall v \in K, \end{cases} \tag{2.1}$$

where

$$K = \{v \in H^1(\Omega) : v \geqslant \psi \text{ a.e. in } \Omega, \ v = g \text{ on } \partial\Omega\}, \tag{2.2}$$

$$\begin{cases} a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + u \cdot v) dx, \\ \langle f, v \rangle = \int_\Omega f \cdot v \, dx \end{cases} \tag{2.3}$$

and $f \in L^2(\Omega)$, $g, \psi \in H^2(\Omega)$, $g \geqslant \psi$ on $\partial\Omega$.

We now solve problem (2.1) using the linear nonconforming finite element approximation. Let B_i $(1 \leqslant i \leqslant 3)$ be the midpoints of edges of triangle $\tau \in \mathscr{T}_h$, and X_h be a space consisting of the piecewise linear functions with nodes at B_i, which is Crouzeix-Raviart's element space $(r=1)$. Let K_h be a convex subset of X_h as follows:

$$K_h = \{v^h \in X_h : v^h \geqslant \psi \text{ at the nodes in } \Omega^h, \ v^h = g(P_m) \text{ at the nodes } m \text{ on } \partial\Omega^h\}, \tag{2.4}$$

where P_m is the intersection point of $\partial\Omega$ with the outer normal at the node m on $\partial\Omega^h$ (Fig. 1).

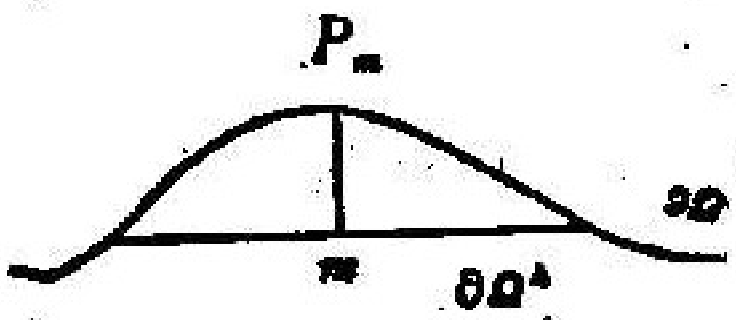

Fig. 1

The linear nonconforming finite element approximation of (2.1) is the following: