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# A RECURSIVE ALGORITHM FOR COMPUTING THE WEIGHTED MOORE-PENROSE INVERSE $A^+_{MN}$*

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**Abstract**

In this paper, we give a recursive algorithm for computing the weighted Moore-Penrose inverse $A^+_{MN}$. This method is a generalization of Greville's method for computing Moore-Penrose inverse $A^+$, and the technique of its proof is new. This method suits the weighted least-squares problem.

## § 1. Introduction

Throughout this paper, let $M$ and $N$ be positive definite matrices of order $m$ and $n$ respectively. Let $A\in C^{m\times n}$. Then there is a unique matrix $X\in C^{n\times m}$ satisfying

$$AXA=A,\ XAX=X,\ (MAX)^*=MAX,\ (NXA)^*=NXA. \tag{1.1}$$

This $X$ is called the weighted $M-P$ inverse of $A$, and is denoted by $X=A^+_{MN}$. Especially, when $M=I_m$ and $N=I_n$, the matrix $X$ satisfying (1.1) is called the $M-P$ inverse of $A$, and is denoted by $X=A^+$, i.e., $A^+=A^+_{I_mI_n}$.

In 1960, A famous recursive method for computing the $M-P$ inverse of $A$ was given by Greville[1].

Let $A_k\in C^{m\times k}$ be the submatrix of $A\in C^{m\times n}$ consisting of its first $k$ columns. For $k=2, \cdots, n$ the matrix $A_k$ is partitioned as

$$A_k=[A_{k-1}\ a_k],$$

where $a_k$ is the $k$-th column of $A$. For $k=2, \cdots, n$ the vectors $d_k$ and $c_k$ are defined by $d_k=A^+_{k-1}a_k$ and $c_k=a_k-A_{k-1}d_k=(I-A_{k-1}A^+_{k-1})a_k$. Then, the $M-P$ inverse of $A_k$ is

$$A^+_k=\begin{pmatrix} A^+_{k-1}-d_kb^*_k \\ b^*_k \end{pmatrix},$$

where

$$b^*_k=\begin{cases}(c^*_kc_k)^{-1}c^*_k & \text{if } c_k\neq 0,\\ (1+d^*_kd_k)^{-1}d^*_kA^+_{k-1} & \text{if } c_k=0.\end{cases}$$

In [2, 3, 4] three different proofs for Greville's method were presented. Greville's method is natural in some applications, for example, the least-squares polynomial approximation problem, regression analysis, etc[5].

There are many formulas for computing the weighted $M-P$ inverse $A^+_{NM}$[6], but they are very complex. In this paper, we will give a recursive algorithm for computing $A^+_{MN}$. This method is a generalization of Greville's method, and the technique of its proof is new. This method suits the weighted least-squares problem.

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## § 2. Preliminaries

In this section we will give three lemmas.

**Lemma 1.** *Let* $A \in C_r^{m\times n}$, $X = A_{MN}^+$; then

(i) $R(X) = N^{-1}R(A^*)$, $N(X) = M^{-1}N(A^*)$,

$R(X^*) = MR(A)$, $N(X^*) = MN(A)$,

$AX = P_{R(A), M^{-1}N(A^*)}$, $XA = P_{N^{-1}R(A^*), N(A)}$.

(ii) $AX = A(A^*MA)^+A^*M$, $XA = N^{-1}A^*(AN^{-1}A^*)^+A$.

(iii) $(A_{MN}^+)^* = (A^*)_{N^{-1}M^{-1}}^+$.

(iv) *Let* $U \in C_{m-r}^{m\times(m-r)}$ *and* $V \in C_{n-r}^{n\times(n-r)}$ *such that* $A^*U = 0$ *and* $AV = 0$; *then*

$I - XA = V(V^*NV)^{-1}V^*N$, $I - AX = M^{-1}U(U^*M^{-1}U)^{-1}U^*$.

*Proof.* (i) See [1, chap. 3].

(ii) and (iii) See [2, chap. 3].

(iv) By hypothesis, $R(U) = N(A^*)$ and $R(V) = N(A)$. Since $V^*NV$ is p.d., inverse $(V^*NV)^{-1}$ exists and is also p.d. Set $V(V^*NV)^{-1}V^*N = E$. Then $E$ is idempotent, and $R(E) = R(V) = N(A)$ and $N(E) = N(V^*N) = N^{-1}N(V^*) = N^{-1}R(A^*)$. Hence $V(V^*NV)^{-1}V^*N = P_{N(A), N^{-1}R(A^*)} = I - P_{N^{-1}R(A^*), N(A)} = I - XA$. A similar argument shows $I - AX = M^{-1}U(U^*M^{-1}U)^{-1}U^*$.

**Lemma 2.** *Let* $A \in C_r^{m\times n}$, $U \in C_{m-r}^{m\times(m-r)}$ *and* $V \in C_{n-r}^{n\times(n-r)}$ *such that*

$$A^*U = 0 \text{ and } AV = 0. \tag{2.1}$$

*Then*

(i)
$$\begin{pmatrix} A & M^{-1}U \\ V^*N & 0 \end{pmatrix} \text{ is nonsingular.} \tag{2.2}$$

(ii)
$$\begin{pmatrix} A & M^{-1}U \\ V^*N & 0 \end{pmatrix}^{-1} = \begin{pmatrix} A_{MN}^+ & V(V^*NV)^{-1} \\ (U^*M^{-1}U)^{-1}U^* & 0 \end{pmatrix}. \tag{2.3}$$

*Proof.* Set $X = A_{MN}^+$. From Lemma 1, we have

$$AX + M^{-1}U(U^*M^{-1}U)^{-1}U^* = AX + (I - AX) = I \tag{2.4}$$

and from (2.1)

$$AV(V^*NV)^{-1} = 0. \tag{2.5}$$

Since $V^*NXA = V^*(NXA)^* = V^*A^*X^*N = 0$,

$$V^*NX = V^*NXAX = 0 \tag{2.6}$$

and, obviously

$$V^*NV(V^*NV)^{-1} = I. \tag{2.7}$$

Using (2.4)—(2.7), we may obtain (2.2) and (2.3) immediately.

**Lemma 3.** *Let* $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ *be a partitioned matrix which is nonsingular, and let the submatrix* $A_{11}$ *also be nonsingular. Then*

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{2.8}$$

*where*

$$B_{11} = A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}A_{21}A_{11}^{-1}, \tag{2.9}$$