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MULTIGRID METHOD FOR ELASTICITY PROBLEMS*

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§ 1. Introduction

The multigrid method is a fast iterative method developed during the sixties for solving elliptic partial differential equations with boundary value condition. Its difference from other iteration methods is that the convergence rate is independent of the grid size h, and the operating amount for obtaining the approximate solutions with the same accuracy is $O(N)$ where N is the number of the unknowns in equations. Since the seventies the MG method has been widely used in many other problems successfully, including some elliptic variational inequalities, whose physical background includes some fluid flow through porous medium, and the water cone problem of an oil well[3,8,9]. This paper discusses how the MG method is used in some elastic mechanical problems, including general two- or three-dimensional elasticity problems and a type of variational inequality of elasticity problems, whose physical background is elastic-rigid and elastic-elastic contact problems without friction. We will prove the convergence of the method and give some numerical examples.

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§ 2. Mathematical Problems and MG Method

Consider an elastic body Ω and a rigid body Ω_1 and suppose that $x=(x_1, x_2, x_3)$ is a set of Cartesian coordinates of a point in the space, $u(x)=(u_1(x), u_2(x), u_3(x))$ is its displacement function, ε_{ij}, σ_{ij} are tensors of strain and stress respectively, $f=(f_1, f_2, f_3)$ is a vector of the body force, $p=(p_1, p_2, p_3)$ is a vector of the surface force, the boundary of Ω, $\partial\Omega=\Gamma_u\cup\Gamma_\sigma\cup\Gamma_c$. On Γ_u and Γ_σ displacements and surface forces are prescribed respectively. Γ_c is the boundary where Ω and Ω_1 are contacting or will contact. So Γ_c is unknown in the problem and is assumed to be a smooth surface. $n=(n_1, n_2, n_3)$ is a unit external normal vector of $\partial\Omega$. Now we consider two types of problems: (I) linear problems, (II) nonlinear problems.

Problem (I.1). Find a displacement function $u(x)$, satisfying the following relations:

$$\varepsilon_{ij}(u)=\varepsilon_{ji}(u)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{2.1}$$

$$\sigma_{ij}(u)=\sigma_{ji}(u)=\lambda\sum_{k=1}^{3}\varepsilon_{kk}(u)\delta_{ij}+2\mu\varepsilon_{ij}(u), \tag{2.2}$$

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$$\sigma_{ij,j}+f_i=0, \quad x\in\Omega,\ i,\ j=1,\ 2,\ 3, \tag{2.3}$$

$$\sigma_{ij}n_j=p_i, \quad x\in\Gamma_\sigma, \tag{2.4}$$

$$u_i=\bar{u}_i, \quad x\in\Gamma_u, \tag{2.5}$$

where δ_{ij} is Kronecker's symbol, and λ, μ are two constants which are assumed to satisfy $\lambda>0$, $\mu>0$. Conventionally, repeated subscript i denotes summation for $i=1, 2, 3$.

Problem (II.1). Find a displacement vector $u(x)$ and the contact boundary Γ_c, satisfying (2.1)—(2.5) and relations:

$$\left.\begin{array}{l} u_in_i-s\leqslant 0,\ \sigma_n\leqslant 0 \\ \sigma_n\cdot(u_in_i-s)=0 \end{array}\right\} \text{ on } \Gamma_c, \tag{2.6}$$

where s is the initial gap between Ω and Ω_1, σ_n is normal stress

$$\sigma_n=\sigma_{ij}n_in_j. \tag{2.7}$$

We define the function space H and its convex subset K

$$H=\{u=(u_1,\ u_2,\ u_3)\,|\,u_i\in H^1(\Omega),\ u_i=\bar{u}_i,\ x\in\Gamma_u\}, \tag{2.8}$$

$$K=\{u\in H,\ u_in_i-s\leqslant 0,\ x\in\Gamma_c\}. \tag{2.9}$$

The norm in H is

$$\|v\|_1=\left\{\int_\Omega (v_iv_i+v_{i,j}v_{i,j})dx\right\}^{1/2}. \tag{2.10}$$

Define a bi-linear functional and a linear functional as follows:

$$a(u,\ v)=\int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v)dx, \tag{2.11}$$

$$f(v)=\int_\Omega f_iv_i\,dx+\int_{\Gamma_\sigma} p_iv_i\,ds. \tag{2.12}$$

According to the Korn inequality, $a(u,\ v)$ is H-coercive, i.e. there exists $\alpha>0$ such that

$$a(u,\ v)\geqslant\alpha\|v\|_1^2, \quad \forall v\in H. \tag{2.13}$$

Problems (I.1), (II.1) have the following equivalent variational problems[6,7]:

Problem (I.2).

$$\begin{cases} \text{Find } u\in H \quad \text{such that} \\ a(u,\ v)=f(v), \quad \forall v\in H. \end{cases} \tag{2.14}$$

Problem (I.3).

$$\begin{cases} \text{Find } u\in H \quad \text{such that} \\ J(u)\leqslant J(v), \quad \forall v\in H, \end{cases} \tag{2.15}$$

where $J(v)=\frac{1}{2}a(v,\ v)-f(v)$.

Problem (II.2).

$$\begin{cases} \text{Find } u\in K \quad \text{such that} \\ a(u,\ v-u)\geqslant f(v-u), \quad \forall v\in K. \end{cases} \tag{2.16}$$

Problem (II.3).

$$\begin{cases} \text{Find } u\in K \quad \text{such that} \\ J(u)\leqslant J(v), \quad \forall v\in K. \end{cases} \tag{2.17}$$